MATH 570: MATHEMATICAL LOGIC

HOMEWORK 12

Due date: Dec 3 (Wed)

1. Let $\mathbf{M}$ be a λ-saturated τ-structure. Prove that if

$$B = \bigcap_{i \in I} C_i = \bigcup_{j \in J} D_j,$$

for some definable (with parameters) sets $C_i, D_j \subseteq M^n$ and $|I|, |J| < \lambda$, then there exists finite $I_0 \subseteq I, J_0 \subseteq J$ such that

$$B = \bigcap_{i \in I_0} C_i = \bigcup_{j \in J_0} D_j.$$

2. Prove the Compactness Theorem using ultraproducts as follows. Let T be a set of τ-sentences and suppose that it is finitely satisfiable, i.e. every finite subset has a model. To warm up, first assume T is countable and construct a model of T as an ultraproduct over any nonprincipal ultrafilter on T. For the general case, let $I = \mathscr{P}_{\text{fin}}(T)$ be the set of all finite subsets of T and take an ultrafilter on I that contains all of the *cones*, i.e. the sets of the form $C_\phi := \{F \in I : \phi \in F\}$, for $\phi \in T$.

3. For a ring R, let $\text{Spec}(R)$ denote the set of prime ideals of R and endow it with the Zariski topology, which is given by declaring the following sets (cones) closed: $C_I = \{J \in \text{Spec}(R) : J \supseteq I\}$, where $I \subseteq R$ is an ideal (not necessarily prime).

(a) First, explain why this indeed defines a topology.

(b) Next, let $\mathbf{K}$ be an algebraically closed field and let $F \subseteq K$ be a universe of a subfield. To every n-type $\mathfrak{p}(\vec{x}) \in S^n_{\mathbf{K}}(F)$, assign the set $I_{\mathfrak{p}} := \{f \in F[\vec{x}] : \text{“}f(\vec{x}) = 0\text{”} \in \mathfrak{p}(\vec{x})\}$ of polynomials over F. Show that $I_{\mathfrak{p}}$ is a prime ideal in $F[\vec{x}]$.

(c) Show that the map $\mathfrak{p} \mapsto I_{\mathfrak{p}}$ is a continuous bijection. Deduce that the Zariski topology is compact.

Hint: For the surjectivity of $\mathfrak{p} \mapsto I_{\mathfrak{p}}$, you may use without proof that for any $J \in \text{Spec}(F[\vec{x}])$, there is $J' \in \text{Spec}(K[\vec{x}])$ such that $J' \cap F[\vec{x}] = J$.

(d) However, show that the Zariski topology is not Hausdorff by noticing that any nonempty Zariski open set must contain the zero ideal. Thus, the map $\mathfrak{p} \mapsto I_{\mathfrak{p}}$ is not a homeomorphism.