MATH 570: MATHEMATICAL LOGIC

HOMEWORK 11

Due date: Nov 19 (Wed)

1. Call a set $A \subseteq \mathbb{Z}$ a *difference set* or a Δ-*set* if A is infinite and is of the form $\{x_n - x_m : n > m\}$ for some sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}$ (possibly with repetitions). Call $C \subseteq \mathbb{Z}$ a Δ^*-set if it intersects every Δ-set; denote by Δ^* the collection of all Δ^*-sets.

(a) Show that Δ-sets enjoy the *Ramsey property*; namely, for any $A, B \subseteq \mathbb{Z}$, whenever $A \cup B$ contain a Δ-set, then at least one of A, B contains a Δ-set.

Hint: Use the infinite Ramsey theorem.

(b) Conclude that Δ^* is a filter.

2. Let X be a topological space and α an ultrafilter on X. Call $x \in X$ a *limit point* of α if every open neighborhood of x belongs to $\mathcal{U}$. Prove the following characterizations of Hausdorffness and compactness.

(a) X is Hausdorff if and only if every ultrafilter on X has at most one limit point.

(b) X is compact if and only if every ultrafilter on X has at least one limit point.

Hint: For $\Rightarrow$, prove the contrapositive, and for $\Leftarrow$, show that any collection of closed sets with the finite intersection property has a nonempty intersection.