MATH 570: MATHEMATICAL LOGIC

HOMEWORK 10

Due date: Nov 12 (Wed)

1. We start by recalling the definition of the arithmetical hierarchy: for a class Γ of subsets of $\mathbb{N}^{<\infty}$, so that each set in Γ is a subset of $\mathbb{N}^k$ for some $k \geq 1$, let

$$\exists^{\mathbb{N}}\Gamma := \{\exists y R(\vec{x}, y) : R(\vec{x}, y) \in \Gamma\},$$

$$\forall^{\mathbb{N}}\Gamma := \{\forall y R(\vec{x}, y) : R(\vec{x}, y) \in \Gamma\},$$

$$\neg\Gamma := \check{\Gamma} := \{\neg R(\vec{x}) : R(\vec{x}) \in \Gamma\}.$$

Assuming the class Σ^0_n is defined, put $\Pi^0_n := \neg\Sigma^0_n$, $\Sigma^0_{n+1} := \exists^{\mathbb{N}}\Pi^0_n$, and $\Delta^0_n := \Sigma^0_n \cap \Pi^0_n$.

(a) Show that $\Delta^0_n \subsetneq \Sigma^0_n \subsetneq \Delta^0_{n+1}$ and $\Delta^0_n \subsetneq \Pi^0_n \subsetneq \Delta^0_{n+1}$. Make sure to also show that inclusions, not just their strictness.

(b) Show that $\bigcup_n \Sigma^0_n$ is precisely the class of all arithmetical sets.

(c) Conclude Tarski's theorem that $\ulcorner \mathsf{Th}(\mathbf{N}) \urcorner$ is not arithmetical.

2. Show that $(\mathbb{N}, 0, S)$ admits effective quantifier elimination. Conclude that the only definable sets are the finite and cofinite sets.

3. Let $\tau = (0, 1, +, -, \cdot, <)$ and $\mathbf{Q} = (\mathbb{Q}, 0, 1, +, -, \cdot, <)$.

(a) Show that for every subset $S \subseteq \mathbb{Q}$ that is definable in $\mathbf{Q}$ by a quantifier free formula, there is $q \in \mathbb{Q}$ such that $(q, \infty) \subseteq S$ or $(q, \infty) \cap S = \varnothing$.

Hint: Prove this by induction on the construction (length) of the quantifier-free formula defining S.

(b) Use (a) to show that $\mathsf{Th}(\mathbf{Q})$ does NOT admit quantifier elimination.

4. Show that the theory of vector spaces over $\mathbb{Q}$ is diagram-complete and hence admits q.e. Conclude that the only definable sets in any $\mathbb{Q}$-vector space are the finite and cofinite sets.

5. Show that a theory is model-complete if and only if for every formula $\phi(\vec{x})$ there is an existential formula $\psi(\vec{x})$ such that $T \vDash [\phi(\vec{x}) \leftrightarrow \psi(\vec{x})]$. Conclude that the same is true with "existential" replaced with "universal".

Hint: For $\Rightarrow$ direction, consider

$$\Gamma(\vec{x}) := \{\psi(\vec{x}) \ \tau\text{-formula} : T \vDash [\phi(\vec{x}) \to \psi(\vec{x})] \text{ and } T \vDash [\psi(\vec{x}) \leftrightarrow \theta(\vec{x})] \text{ for some existential } \tau\text{-formula } \theta(\vec{x})\},$$

and assuming for contradiction that $T \cup \Gamma(\vec{x}) \cup \{\neg\phi(\vec{x})\}$ has a model $\mathbf{A}$ with $\vec{a} := \vec{x}^{\mathbf{A}}$, show that $T \cup \mathsf{Diag}(\mathbf{A}, A) \cup \{\phi(\vec{a})\}$ also has a model $\mathbf{B}$.