MATH 570: MATHEMATICAL LOGIC

HOMEWORK 1

Due on Wednesday, Sep 3

1. Define appropriate signatures for
(a) vector spaces over $\mathbb{Q}$;
(b) metric spaces.

2. (a) Show by an example that in the signature $\tau_{\text{group}} = (1, \cdot)$, a substructure of a group need not be a subgroup.
(b) Define a signature for groups different from the one above so that a substructure of a group is a subgroup.

3. If $h : \mathbf{A} \to \mathbf{B}$ is a τ-homomorphism then the image $h(A)$ is a universe of a substructure of $\mathbf{B}$.

4. Prove that if τ does not contain any relation symbols, then any bijective τ-homomorphism is a τ-isomorphism. (That's why this happens with groups and rings, but not with graphs or orderings.)

5. A structure is called *rigid* if it has no automorphisms[1] other than the identity. Show that the structures $\mathbf{N} = (\mathbb{N}, 0, S, +, \cdot)$ and $\mathbf{Q} = (\mathbb{Q}, 0, 1, +, \cdot)$ are rigid.

6. A structure $\mathbf{M}$ is called *ultrahomogeneous* if given any isomorphism between two finitely generated substructures, it extends to an automorphism of the whole structure, i.e. if $\mathbf{A}, \mathbf{B}$ are finitely generated substructures of $\mathbf{M}$ and $h : \mathbf{A} \to \mathbf{B}$ is an isomorphism, then there is an automorphism $\bar{h}$ of $\mathbf{M}$ with $\bar{h} \sqsupseteq h$. Show that $(\mathbb{Q}, <)$ is ultrahomogeneous. The same proof should also work to show that $(\mathbb{R}, <)$ is ultrahomogeneous.

7. For a signature τ, a *τ-sentence* is a τ-formula that does not have free variables. For τ-structures $\mathbf{A}, \mathbf{B}$, we write $\mathbf{A} \equiv \mathbf{B}$ if for every τ-sentence ϕ, $\mathbf{A} \models \phi \iff \mathbf{B} \models \phi$.
(a) Show that there is a τ_{group}-sentence ϕ such that

$$\mathbf{M} \models \phi \iff \mathbf{M} \simeq \mathbb{Z}/2\mathbb{Z}.$$

(b) More generally, let τ be a finite signature and $\mathbf{A}$ be a finite τ-structure. Show that there is a τ-sentence ϕ such that for any τ-structure $\mathbf{B}$,

$$\mathbf{B} \models \phi \iff \mathbf{B} \simeq \mathbf{A}.$$

In particular,

$$\mathbf{B} \equiv \mathbf{A} \iff \mathbf{B} \simeq \mathbf{A}.$$

[1] isomorphism with itself