QUIZ 3

a. By the cosine law applied to ABP & ACP, we have (respectively):

$$c^2 = d^2 + m^2 - 2dm\cos(\angle APB), \qquad b^2 = d^2 + n^2 - 2dn\cos(\angle APC).$$

But, we know that $\cos(\angle APB) + \cos(\angle APC) = 0$ since $\angle APB + \angle APC = 2\pi$. Consequently,

$$c^2 n + b^2 m = (d^2 + m^2 - 2dm\cos(\angle APB))n + (d^2 + n^2 - 2dn\cos(\angle APC))m$$
$$= d^2 n + m^2 n + d^2 m + n^2 m - 2dmn(\cos(\angle APB) + \cos(\angle APC))$$
$$= (m+n)(d^2 + mn)$$

as was required.

b. If $P = A'$ (bisector of BC), we have by a)

$$b^2 \cdot a/2 + c^2 \cdot a/2 = (a/2 + a/2)\cdot(|AA'|^2 + \tfrac{a}{2}\cdot\tfrac{a}{2}) \Rightarrow \tfrac{1}{2}a(b^2+c^2) = a(|AA'|^2 + \tfrac{a^2}{4}) \Rightarrow b^2 + c^2 = 2|AA'|^2 + a^2/2$$

and by applying to $P = B'$, $P = C'$ we get the analogous equations: $a^2 + c^2 = 2|BB'|^2 + b^2/2$ and $a^2 + b^2 = 2|CC'|^2 + c^2/2$.
Recall that $|AG| = 2|AG'|$ or that $|AG| = \frac{2}{3}|AA'|$. This shows that, summing the three previous eqn,

$$(b^2+c^2) + (a^2+c^2) + (a^2+b^2) = 2(|AA'|^2 + |BB'|^2 + |CC'|^2) + \frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2}$$
$$\Rightarrow \frac{3}{2}(a^2+b^2+c^2) = 2(|AA'|^2 + |BB'|^2 + |CC'|^2) = 2\cdot\frac{9}{4}(|AG|^2 + |BG|^2 + |CG|^2)$$
$$\Rightarrow a^2+b^2+c^2 = \underbrace{\left(\frac{2}{3}\cdot 2\cdot\frac{9}{4}\right)}_{=3}(|AG|^2 + |BG|^2 + |CG|^2)$$

as required.