QUIZ 2 SOLUTIONS

1. Let $f(\vec{x}) = \dfrac{1}{(x_1^2+\dots+x_n^2)^{n/2-1}} = (x_1^2+\dots+x_n^2)^{1-n/2}$. By the chain rule,

$$\frac{\partial f}{\partial x_i} = (1-n/2)(x_1^2+\dots+x_n^2)^{-n/2}\cdot 2x_i = (2-n)\, x_i\,(x_1^2+\dots+x_n^2)^{-n/2}$$

and again, by applying the chain rule and Leibniz rule,

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left((2-n)\,x_i\,(x_1^2+\dots+x_n^2)^{-n/2}\right) = (2-n)\left((x_1^2+\dots+x_n^2)^{-n/2} + x_i\left(\tfrac{-n}{2}\right)(x_1^2+\dots+x_n^2)^{-n/2-1}\cdot 2x_i\right)$$

$$= (2-n)(x_1^2+\dots+x_n^2)^{-n/2}\left(1 - n x_i^2\,(x_1^2+\dots+x_n^2)^{-1}\right)$$

Consequently,

$$\sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i^2} = \sum_{i=1}^{n}(2-n)(x_1^2+\dots+x_n^2)^{-n/2}\left(1-n x_i^2(x_1^2+\dots+x_n^2)^{-1}\right)$$

$$= (2-n)(x_1^2+\dots+x_n^2)^{-n/2}\cdot\left(\sum_{i=1}^{n} 1 - n x_i^2 (x_1^2+\dots+x_n^2)^{-1}\right)$$

$$= (2-n)(x_1^2+\dots+x_n^2)^{-n/2}\cdot\left(\frac{\sum_{i=1}^{n}\left((x_1^2+\dots+x_n^2) - n x_i^2\right)}{x_1^2+\dots+x_n^2}\right)$$

$$= (2-n)(x_1^2+\dots+x_n^2)^{-n/2}\cdot\left(\frac{\underbrace{n(x_1^2+\dots+x_n^2) - n\sum_{i=1}^{n} x_i^2}_{=0}}{x_1^2+\dots+x_n^2}\right) = 0$$

as requested.

2. Let $f(x,y) = 3xy - x^2y - xy^2$. Then,

$$\nabla f = (3y - 2xy - y^2,\ 3x - x^2 - 2xy) = (y(3-2x-y),\ x(3-2y-x))$$

which shows that the critical pts are solutions to the system

$$\begin{cases} y(3-2x-y) = 0 \\ x(3-2y-x) = 0 \end{cases} \Rightarrow (x,y) = (0,0),(3,0),(0,3) \text{ or } \left(\begin{cases} 2x+y=3 \\ x+2y=3\end{cases} \Rightarrow \begin{cases} x+y=2 \\ x-y=0\end{cases} \Rightarrow (x,y)=(1,1)\right).$$

The Hessian matrix of f is $H(f) = \begin{pmatrix} -2y & 3-2x-2y \\ 3-2x-2y & -2x\end{pmatrix}$, which evaluates to

(x,y)	$(0,0)$	$(3,0)$	$(0,3)$	$(1,1)$
$H(f)$	$\begin{pmatrix} 0 & 3 \\ 3 & 0\end{pmatrix}$	$\begin{pmatrix} 0 & -3 \\ -3 & -6\end{pmatrix}$	$\begin{pmatrix} 0 & -3 \\ -3 & -6\end{pmatrix}$	$\begin{pmatrix} -2 & -1 \\ -1 & -2\end{pmatrix}$
$\det H(f)$	-9	-9	-9	3
	saddle pt	saddle pt	saddle pt	min/max → is a max since $\frac{\partial^2 f}{\partial x^2} = -2 < 0$