practice final

math222, calculus III
summer 2025

Justify all your claims rigorously. This exam is worth 55% of the grade for math222 and contains 8 questions. Allotted time is 3 hours.
This is a closed book exam, no material (e.g. calculator, dictionary, crib sheet) is permitted except pen/pencil and eraser.

---

useful formulas.

curves
Let $\vec{r}:(a,b)\to\mathbb{R}^3$ be a regular curve having no singular point of order 1. Then, the curvature κ and the torsion τ satisfy

$$\kappa(t)=\frac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}\qquad \tau(t)=\frac{(\vec{r}''(t)\times\vec{r}'(t))\bullet\vec{r}'''(t)}{\|\vec{r}'(t)\times\vec{r}''(t)\|^2}$$

When $\vec{r}$ is parametrized by arc-length, the Frenet-Serret equations are

$$\begin{aligned}\vec{t}'(s)&=\kappa(s)\vec{n}(s)\\ \vec{n}'(s)&=-\kappa(s)\vec{t}(s)-\tau(s)\vec{b}(s)\\ \vec{b}'(s)&=\tau(s)\vec{n}(s)\end{aligned}$$

where $\vec{t}(s)$ is the tangent vector, $\vec{n}(s)$ is the normal vector and $\vec{b}(s)$ is the binormal vector.

coordinate systems

— The polar coordinates in $2d$ are given by

$$x(r,\theta)=r\cos(\theta)\qquad y(r,\theta)=r\sin(\theta)$$

where $r\in\mathbb{R}_{\geq 0}$ and $\theta\in[0,2\pi)$.

— The spherical coordinates in $3d$ are given by

$$x(\rho,\theta,\phi)=\rho\cos(\theta)\sin(\phi)\qquad y(\rho,\theta,\phi)=\rho\sin(\theta)\sin(\phi)\qquad z(\rho,\theta,\phi)=\rho\cos(\phi)$$

where $\rho\in\mathbb{R}_{\geq 0}$, $\theta\in[0,2\pi)$ and $\phi\in[0,\pi]$.

---

1. Compute the Taylor series centred at 2 of the function $f(x)=\ln(1+x)$ and find its interval of convergence.

2. Evaluate the following integrals :

a. $\displaystyle\int_0^1\int_{3y}^3 e^{x^2}\,\mathrm{d}x\mathrm{d}y$

b. $\displaystyle\int_0^\pi\int_0^x\int_0^{x+z}\sin(x-2y+z)\,\mathrm{d}y\mathrm{d}z\mathrm{d}x$

3. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

4. Show that if $w = f(x, y, z)$ and $x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$, then

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial^2 z} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial w}{\partial\rho}\right) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 w}{\partial\phi^2}.$$

5. Consider the two disks whose boundaries are given respectively by the polar equations

$$r = \cos\theta,\ \ \theta \in [-\pi/2, \pi/2] \qquad \text{and} \qquad r = \sin\theta,\ \ \theta \in [0, \pi].$$

Compute the area of the overlap of the two disks.

6. Find the critical points of the function

$$f(x, y) = (6x - 2x^2)(4y - y^2)$$

and determine if the critical points are minima, maxima or saddle points.

7. Let $\vec{r} : (a, b) \to \mathbb{R}^2$ be a regular parametrized curve whose curvature $\kappa(t)$ never vanishes. The *evolute* of $\vec{r}$ is the curve $\overrightarrow{r_{\text{ev}}} : (a, b) \to \mathbb{R}^2$ defined by the parametric equation

$$\overrightarrow{r_{\text{ev}}}(t) = \vec{r}(t) + \frac{1}{\kappa(t)}\vec{n}(t)$$

where $\vec{n}(t)$ is the normal vector of $\vec{r}(t)$. Recall that the tangent and normal vectors to a curve which is not necessarily parametrized by arc-length are given by

$$\vec{t}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \qquad \text{and} \qquad \vec{n}(t) = \frac{\vec{t}\,'(t)}{\|\vec{t}\,'(t)\|}.$$

Compute the evolute of the curve $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ given by

$$\vec{r}(t) = (t, \cosh(t))$$

where $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$.

8. Let B be a ball of radius $a > 0$. Find the volume of the portion of B lying between the half-cones whose equations are given in spherical coordinates by $\phi = \pi/4$ and $\phi = \pi/3$.