# midterm SOLUTIONS

1.a) Limit comparison test with $\sum \frac{1}{n^3}$ (p-series with $p>1 \Rightarrow$ convergent)

$$\lim_{n\to\infty} \frac{\frac{n^2+2n+3}{3n^5+2n+1}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^5+2n^4+3n^3}{3n^5+2n+1} = \frac{1}{3}$$

and series have the same behavior $\Rightarrow$ converges.

b) Cauchy's condensation test:

$\sum_{k=2}^{+\infty} \frac{1}{(\ln n)^k}$ converges if and only if $\sum_{n=2}^{+\infty} 2^n \frac{1}{(\ln 2^n)^k}$ converges

but,

$$\sum_{n=2}^{+\infty} 2^n \frac{1}{(\ln(2^n))^k} = \sum_{n=2}^{+\infty} 2^n \cdot \frac{1}{n^k \cdot \ln 2}$$

and the main term of this series does not tend to zero ($\lim_{n\to\infty} \frac{2^n}{n^k} = +\infty$).

Thus, the original series diverges.

c) Alternating series test: Converges $\Leftrightarrow$ main term tends to zero. And,

$$\lim_{n\to\infty} \frac{n^2}{n^2+\ln n} = 1$$

$\Rightarrow$ series diverges.

d) This is a geometric series with $|r| = \frac{1}{2} < 1 \Rightarrow$ convergent.

2.a) Root test or ratio test or geometric series: $\left|\frac{-x}{2}\right| < 1 \Leftrightarrow |x|<2 \Rightarrow$ radius $= 2$, centered at $x=0$

On the boundary: $x=-2$ $\sum_{n=1}^{+\infty} 1$ divergent, $x=-2$ $\sum_{n=1}^{+\infty} (-1)^n$ divergent

Interval of convergence is $(-2,2)$.

b) Ratio test: $\lim_{n\to\infty} \left|(x-5)\cdot\frac{n\,2^n}{(n+1)\,2^{n+1}}\right| \longrightarrow \frac{1}{2}|x-5| \Rightarrow$ radius $= 2$, centered at $x=5$.

On the boundary: $x=3$ $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ convergent (seen in class), $x=7$ $\sum_{n=1}^{+\infty} \frac{1}{n}$ divergent (seen in class)

Interval of convergence is $[3,7)$

c) Root test: $\lim_{n\to\infty} \sqrt[n]{n^n |x+\pi|^n} = \lim_{n\to\infty} |x+\pi| \cdot n \longrightarrow \begin{cases} 0 & \text{if } x=-\pi \\ +\infty & \text{otherwise} \end{cases} \Rightarrow$ radius $=0$, centered at $x=-\pi$.

Interval of convergence is $\{-\pi\}$.

d) Ratio test: $\lim_{n\to\infty} |x| \cdot \frac{n \ln(n)}{n+1 \ln(n+1)} \to |x| \Rightarrow$ radius $=1$, centered at $x=0$

On the boundary: $x=-1$ $\sum \frac{1}{n \ln n}$ divergent (seen in class), $x=1$ $\sum \frac{(-1)^n}{n \ln n}$ convergent (alt series test).

Interval of convergence is $[-1,1]$.

3. a) Direct computation:

$$\left.\begin{aligned} f(x) &= \tan x \\ f^{(1)}(x) &= \sec^2 x \\ f^{(2)}(x) &= 2\sec^2 x \tan x \\ f^{(3)}(x) &= 2\sec^4 x + 4\sec^2 x \tan^2 x \end{aligned}\right] \begin{matrix}\text{evaluate} \\ \text{at} \\ x=\pi/4 \\ \to\end{matrix} \quad \begin{aligned} f(\pi/4) &= 1 \\ f^{(1)}(\pi/4) &= 2 \\ f^{(2)}(\pi/4) &= 4 \\ f^{(3)}(\pi/4) &= 16 \end{aligned}$$

Taylor poly is

$$T_3(x) = 1 + 2(x-\pi/4) + \frac{4}{2!}(x-\pi/4)^2 + \frac{16}{3!}(x-\pi/4)^3$$

b) Direct computation:

$$\left.\begin{aligned} f(x) &= x^2 e^x \\ f^{(1)}(x) &= e^x(2x+x^2) \\ f^{(2)}(x) &= e^x(2+4x+x^2) \\ f^{(3)}(x) &= e^x(6+6x+x^2) \\ f^{(4)}(x) &= e^x(12+8x+x^2) \end{aligned}\right] \begin{matrix}\text{evaluate} \\ \text{at } x=1 \\ \to\end{matrix} \quad \begin{aligned} f(1) &= e \\ f^{(1)}(1) &= 3e \\ f^{(2)}(1) &= 7e \\ f^{(3)}(1) &= 13e \\ f^{(4)}(1) &= 21e \end{aligned}$$

Taylor poly is

$$T_4(x) = e + 3e(x-1) + \frac{7e}{2!}(x-1)^2 + \frac{13e}{3!}(x-1)^3 + \frac{21e}{4!}(x-1)^4$$

4. Using partial fractions

$$\frac{1}{n(n+k)} = \frac{A}{n} + \frac{B}{n+k} \Rightarrow 1 = A(n+k) + Bn \Rightarrow A = \frac{1}{k},\ B = -1/k$$

For $N \geqslant k$, we have

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+k)} = \sum_{n=1}^{N} \frac{1}{k}\cdot\frac{1}{n} - \frac{1}{k}\cdot\frac{1}{n+k} = \frac{1}{k}\left(\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{1}{n+k}\right)$$

split the two sums

$$= \frac{1}{k}\left(\sum_{n=1}^{k}\frac{1}{n} + \sum_{n=k+1}^{N}\frac{1}{n} - \sum_{n=1}^{N-k}\frac{1}{n+k} - \sum_{n=N-k}^{N}\frac{1}{n+k}\right)$$

reindex

$$= \frac{1}{k}\left(h_k + \cancel{\sum_{n=k+1}^{N}\frac{1}{n}} - \cancel{\sum_{n=k+1}^{N}\frac{1}{n}} - \sum_{n=N}^{N+k}\frac{1}{n}\right)$$

$$= \frac{1}{k}h_k - \underbrace{\sum_{n=N}^{N+k}\frac{1}{n}}$$

$= \frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{N+k}$ is a sum of $(k+1)$ terms, and all of the are $\leq \frac{1}{N}$

If we consider the sequence $x_N = \sum_{n=N}^{N+k}\frac{1}{n}$, we have $0 \leq x_N \leq (k+1)\cdot\frac{1}{N}$ and this implies $x_N \to 0$.

Thus, $S_N = \frac{1}{k}h_k - x_N \xrightarrow[N\to\infty]{} \frac{1}{k}h_k$ as requested.

5. Direct computation:

$\vec{r}\,'(t) = (-a\sin t, b\cos t, 0) \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$

$\vec{r}\,''(t) = (-a\cos t, -b\sin t, 0)$

$\vec{r}\,'(t) \times \vec{r}\,''(t) = (0, 0, ab)$

$$\kappa(t) = \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$$

Evaluate:

$\kappa(0) = a/b^2 \quad \kappa(\pi/2) = b/a^2 \quad \kappa(\pi) = a/b^2 \quad \kappa(3\pi/2) = b/a^2$.

6.a) Equation of a cone : $x^2+y^2=z^2$ $(a=b=c=1$ here$)$

$$\underbrace{(e^{-2t}\cos t)^2}_{r_x(t)}+\underbrace{(e^{-2t}\sin t)^2}_{r_y(t)} = e^{-4t}(\cos^2 t+\sin^2 t) = \underbrace{(e^{-2t})^2}_{r_z(t)}$$

b) Compute:

$$\vec{r}\,'(t)= e^{-2t}\left(-2\cos(t)-\sin(t),\ \cos(t)-2\sin(t),\ -2\right)$$

$$\|\vec{r}\,'(t)\| = 3e^{-2t}$$

$$s(t) = \int_0^t 3e^{-2\tau}\,d\tau = \left(-\tfrac{3}{2}e^{-2\tau}\right)\Big|_{\tau=0}^{t} = \tfrac{3}{2}\left(1-e^{-2t}\right)$$

$$\Rightarrow\ \tfrac{2}{3}s = 1-e^{-2t}\ \Rightarrow\ e^{-2t} = 1-\tfrac{2}{3}s\ \Rightarrow\ t = -\tfrac{1}{2}\ln\left(1-\tfrac{2}{3}s\right)$$

Notice that here, $s\in[0,3/2)$.

Arc-length param is

$$\vec{r}(s) = \left(1-\tfrac{2}{3}s\right)\cdot\left(\cos\left(-\tfrac{1}{2}\ln\left(1-\tfrac{2}{3}s\right)\right),\ \sin\left(-\tfrac{1}{2}\ln\left(1-\tfrac{2}{3}s\right)\right),\ 1\right)$$

c) Direct computation:

$$\vec{r}\,''(t)= e^{-2t}\left(3\cos(t)+4\sin(t),\ -4\cos(t)+3\sin(t),\ 4\right)$$

$$\vec{r}\,''(t)\times\vec{r}\,'(t) = e^{-4t}\left(4\cos(t)+2\sin(t),\ 4\sin(t)-2\cos(t),\ -5\right)$$

$$\Rightarrow \|\vec{r}\,''(t)\times\vec{r}\,'(t)\| = 3\sqrt{5}\,e^{-4t}$$

and $\|\vec{r}\,'(t)\| = 3e^{-2t}$

$$\kappa(t) = \frac{\|\vec{r}\,''(t)\times\vec{r}\,'(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{\sqrt{5}}{9}e^{-2t}$$

d) Again, direct computation:

$$\vec{r}\,'''(t) = e^{-2t}\left(-2\cos(t)-11\sin(t),\ 11\cos(t)-2\sin(t),\ -8\right)$$

$$\left(\vec{r}\,''(t)\times\vec{r}\,'(t)\right)\cdot\vec{r}\,'''(t) = 10\,e^{-6t}$$

$$\tau(t) = \frac{\left(\vec{r}\,''(t)\times\vec{r}\,'(t)\right)\cdot\vec{r}\,'''(t)}{\|\vec{r}\,''(t)\times\vec{r}\,'(t)\|^2} = \frac{2}{9}e^{2t}$$