midterm
math222, calculus III
summer 2025

Justify all your claims rigorously. This exam is worth 33% of the grade for math222. Allotted time is 2 hours 20 minutes.

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useful formulas.
Let $\vec{r} : (a, b) \to \mathbb{R}^3$ be a regular curve having no singular point of order 1. Then, the curvature κ and the torsion τ satisfy

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} \qquad \tau(t) = \frac{(\vec{r}''(t) \times \vec{r}'(t)) \bullet \vec{r}'''(t)}{\|\vec{r}'(t) \times \vec{r}''(t)\|^2}$$

When $\vec{r}$ is parametrized by arc-length, the Frenet-Serret equations are

$$\begin{aligned} \vec{t}'(s) &= \kappa(s)\vec{n}(s) \\ \vec{n}'(s) &= -\kappa(s)\vec{t}(s) - \tau(s)\vec{b}(s) \\ \vec{b}'(s) &= \tau(s)\vec{n}(s) \end{aligned}$$

where $\vec{t}(s)$ is the tangent vector, $\vec{n}(s)$ is the normal vector and $\vec{b}(s)$ is the binormal vector.

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1. (8 points) Determine whether the following series are convergent or divergent. Specify the convergence criteria you are applying.

- **a.** $\sum_{n=1}^{\infty} \frac{n^2 + 2n + 3}{3n^5 + 2n + 1}$
- **b.** $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^k}$ for $k \in \mathbb{R}_{>0}$ fixed
- **c.** $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2 + \ln n}$
- **d.** $\sum_{n=1}^{\infty} \frac{1}{2^n}$

2. (8 points) Compute the interval of convergence of the following power series.

- **a.** $\sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n} x^n$
- **b.** $\sum_{n=1}^{\infty} \frac{1}{n2^n} (x - 5)^n$
- **c.** $\sum_{n=1}^{\infty} n^n (x + \pi)^n$
- **d.** $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} x^n$

3. (6 points) Recall that, for a differentiable function $f : (c, d) \to \mathbb{R}$, its Taylor polynomial of degree N centred at $a \in (c, d)$ is defined by

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

a. For $f(x) = \tan x$, compute its Taylor polynomial $T_3(x)$ centred at $\pi/4$.

b. For $f(x) = x^2 e^x$, compute its Taylor polynomial $T_4(x)$ centred at 1.

4. (6 points) Fix $k \in \mathbb{Z}_{\geq 1}$ and let $h_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$. Using a telescoping sum, show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+k)} = \frac{h_k}{k}.$$

5. (6 points) Let $a, b \in \mathbb{R}_{>0}$. Consider the curve $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\vec{r}(t) = (a\cos(t), b\sin(t)).$$

Compute the curvature of $\vec{r}$ at $t = 0, \pi/2, \pi, 3\pi/2$.

6. (12 points) Consider the curve $\vec{r} : [0, \infty) \to \mathbb{R}^3$ defined by

$$\vec{r}(t) = (e^{-2t}\cos(t), e^{-2t}\sin(t), e^{-2t}).$$

a. Show that the trace of $\vec{r}$ lies on a cone.

b. Give the arc-length parametrization of $\vec{r}$.

c. Compute the curvature of $\vec{r}$.

d. Compute the torsion of $\vec{r}$.