1. $f(x) = xe^{2x}$

We know the taylor series of e^x :

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \Rightarrow e^{2x} = \sum_{n=0}^{+\infty} \frac{(2x)^n}{n!}$$

Thus, the taylor series of f is

$$xe^{2x} = x \sum_{n=0}^{+\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{+\infty} \frac{2^n x^{n+1}}{n!}$$

By the ratio test, we have

$$\left| \frac{2^{n+1} x^{n+2}}{(n+1)!} \frac{n!}{2^n x^{n+1}} \right| = \frac{2|x|}{n} \xrightarrow[n \to +\infty]{} 0$$

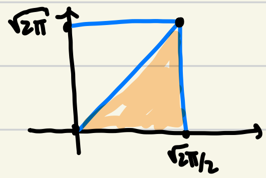
which shows that is has interval of convergence $= (-\infty, +\infty) = \mathbb{R}$

2. a) $\int_0^{\sqrt{2\pi}} \int_{y/2}^{\sqrt{2\pi}/2} \cos(x^2)\, dx\, dy$

domain of integration:

$0 \leq y \leq \sqrt{2\pi}$

$y/2 \leq x \leq \sqrt{2\pi}/2$



$0 \leq x \leq \frac{\sqrt{2\pi}}{2}$

$0 \leq y \leq 2x$

Consequently,

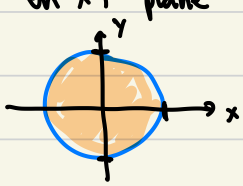
$$\int_0^{\sqrt{2\pi}} \int_{y/2}^{\sqrt{2\pi}/2} \cos(x^2)\, dx\, dy = \int_0^{\sqrt{2\pi}/2} \int_0^{2x} \cos(x^2)\, dy\, dx = \int_0^{\sqrt{2\pi}/2} 2x \cos(x^2)\, dx = \sin(x^2) \Big|_0^{\sqrt{2\pi}/2}$$

$$= \sin(\pi/2) - \sin(0) = 1$$

b) $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{1} xy\, dz\, dx\, dy$

domain of integration: use cylindrical coordinates $x = r\cos\theta$ $y = r\sin\theta$ $z = z$

projection on XY plane



$\theta \in [0, 2\pi)$

$r \in [0,1]$

and region is $\left\{ (r,\theta,z) : \begin{array}{l} \theta \in [0,2\pi) \\ r \in [0,1] \end{array} \; r \leq z \leq 1 \right\}$

Consequently,

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{1} xy\, dz\, dx\, dy = \int_0^{2\pi} \int_0^1 \int_r^1 r\cos\theta\, r\sin\theta\; r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \cos\theta \sin\theta\, d\theta \cdot \int_0^1 \int_r^1 r^3\, dz\, dr$$

and notice that

$$\int_0^{2\pi} \cos\theta \sin\theta\, d\theta = \int_0^{2\pi} \frac{1}{2} \sin(2\theta)\, d\theta = \frac{-1}{4} \cos 2\theta \Big|_0^{2\pi} = \frac{-1}{4}(\cos 4\pi - \cos 0) = 0$$

thus,

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{1} xy\, dz\, dx\, dy = 0 \cdot \int_0^1 \int_r^1 r^3\, dz\, dr = 0$$

3. $f(x,y) = e^{-xy}$ $\qquad \nabla f = (-ye^{-xy}, -xe^{-xy})$

$g(x,y) = x^2 + 4y^2$ $\qquad \nabla g = (2x, 8y)$

Lagrange multipliers says : Solve

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases}$$

and this gives the system

$$\begin{cases} -ye^{-xy} = 2\lambda x & (1) \\ -xe^{-xy} = 8\lambda y & (2) \\ x^2 + 4y^2 = 1 & (3) \end{cases} \quad \begin{matrix} (1)\cdot x \\ \text{and} \\ (2)\cdot y \\ \Rightarrow \end{matrix} \quad \begin{matrix} -xye^{-xy} = 2\lambda x^2 \\ -xye^{-xy} = 8\lambda y^2 \end{matrix} \quad \Rightarrow 2\lambda x^2 = 8\lambda y^2$$

If $\lambda = 0$, eqns (1) and (2) become

$$-ye^{-xy} = 0 \text{ and } xe^{-xy} = 0 \Rightarrow x = y = 0$$

which is incompatible with eqn (3). We deduce that $\lambda \neq 0$ and we have

$$2\lambda x^2 = 8\lambda y^2 \Rightarrow x^2 = 4y^2$$

and substituting in (3) yields

$$2x^2 = 1 \Rightarrow x = \pm\sqrt{2}/2 \Rightarrow y^2 = 1/8 \Rightarrow y = \pm \sqrt{2}/4 .$$

The mins/maxs are given by

$$f(\pm\sqrt{2}/2, \pm\sqrt{2}/4) = e^{-(\pm\sqrt{2}/2)\cdot(\pm\sqrt{2}/4)} = e^{\pm 1/4}$$

$\Rightarrow$ min is $e^{-1/4}$ and max is $e^{1/4}$.

$x = r\cos\theta \qquad x_r = \cos\theta \qquad x_\theta = -r\sin\theta$

$y = r\sin\theta \qquad y_r = \sin\theta \qquad y_\theta = r\cos\theta$

$z = f(x,y)$

first order partials

$z_r = z_x x_r + z_y y_r = \cos\theta\, z_x + \sin\theta\, z_y$

$z_\theta = z_x x_\theta + z_y y_\theta = -r\sin\theta\, z_x + r\cos\theta\, z_y$

second order partials

$$\begin{aligned}
z_{rr} &= \frac{\partial}{\partial r}\left(\cos\theta\, z_x + \sin\theta\, z_y\right) \\
&= \cos\theta\left(z_{xx} x_r + z_{xy} y_r\right) + \sin\theta\left(z_{yx} x_r + z_{yy} y_r\right) \\
&= \cos\theta\left(\cos\theta\, z_{xx} + \sin\theta\, z_{xy}\right) + \sin\theta\left(\cos\theta\, z_{xy} + \sin\theta\, z_{yy}\right) \\
&= \cos^2\theta\, z_{xx} + 2\sin\theta\cos\theta\, z_{xy} + \sin^2\theta\, z_{yy}
\end{aligned}$$

$$\begin{aligned}
z_{\theta\theta} &= \frac{\partial}{\partial\theta}\left(-r\sin\theta\, z_x + r\cos\theta\, z_y\right) \\
&= -r\left(\cos\theta\, z_x + \sin\theta \frac{\partial}{\partial\theta}(z_x)\right) + r\left(-\sin\theta\, z_y + \cos\theta\frac{\partial}{\partial\theta}(z_y)\right) \\
&= -r\left(\cos\theta\, z_x + \sin\theta\, z_y\right) + r\left(-\sin\theta\left(z_{xx} x_\theta + z_{xy} y_\theta\right) + \cos\theta\left(z_{yx} x_\theta + z_{yy} y_\theta\right)\right) \\
&= -r\left(\cos\theta\, z_x + \sin\theta\, z_y\right) + r\left(-\sin\theta\left(-r\sin\theta\, z_{xx} + r\cos\theta\, z_{xy}\right) + \cos\theta\left(-r\sin\theta\, z_{xy} + r\cos\theta\, z_{yy}\right)\right) \\
&= -r\left(\cos\theta\, z_x + \sin\theta\, z_y\right) + r^2\left(\sin^2\theta\, z_{xx} - 2\sin\theta\cos\theta\, z_{xx} + \cos^2 z_{yy}\right)
\end{aligned}$$

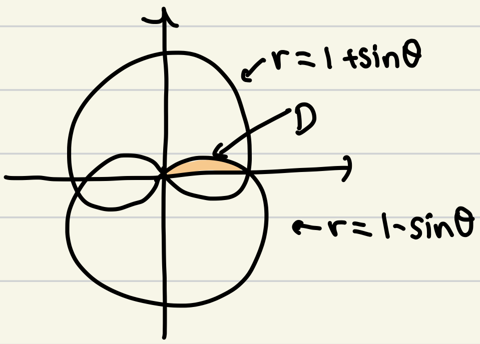
Thus

$$\begin{aligned}
\frac{1}{r^2} z_{\theta\theta} + z_{rr} + \frac{1}{r} z_r &= \cancel{\frac{-1}{r}\left(\cos\theta\, z_x + \sin\theta\, z_y\right)} + \left(\sin^2\theta\, z_{xx} \cancel{- 2\sin\theta\cos\theta\, z_{xx}} + \cos^2 z_{yy}\right) \\
&\quad + \cos^2\theta\, z_{xx} + \cancel{2\sin\theta\cos\theta\, z_{xy}} + \sin^2\theta\, z_{yy} \\
&\quad + \cancel{\frac{1}{r}\left(\cos\theta\, z_x + \sin\theta\, z_y\right)}
\end{aligned}$$

$= \left(\cos^2\theta + \sin^2\theta\right) z_{xx} + \left(\cos^2\theta + \sin^2\theta\right) z_{yy} = z_{xx} + z_{yy}$

5. We compute the area of 1/4 of the shaded region. The domain is

$$D = \{\theta \in [0, \pi/2],\ 0 \le r \le 1 - \sin\theta\}$$



which yields

$$\iint_D dA = \int_0^{\pi/2}\int_0^{1-\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2} \tfrac{1}{2}(1 - 2\sin\theta + \sin^2\theta)\,d\theta$$

$$= \frac{\pi}{4} - (\cos\theta)\Big|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= \pi/4 - 1 + \frac{1}{2}\int_0^{\pi/2} \frac{1}{2}(1 - \cos 2\theta)\,d\theta$$

$$= \pi/4 - 1 + \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= \pi/4 - 1 + \frac{1}{4}\left(\pi/2 + \frac{1}{2}\right)$$

$$= \frac{1}{8}(2\pi - 8 + \pi + 1) = \frac{1}{8}(3\pi - 7)$$

$\sin^2\theta + \cos^2\theta = 1$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\quad = 1 - 2\sin^2\theta$

Thus, total area is $4 \cdot \frac{1}{8}(3\pi - 7) = \frac{1}{2}(3\pi - 7)$

6. $f(x,y) = x^3 - 6xy + 8y^3$

$\nabla f = (3x^2 - 6y,\ 24y^2 - 6x)$

$H(f) = \begin{pmatrix} 6x & -6 \\ -6 & 48y \end{pmatrix}$

First, we find the critical pts: $\nabla f = 0$

$\begin{cases} 3x^2 = 6y \\ 24y^2 = 6x \end{cases} \Rightarrow 9x^4 = 6^2 y^2 = \frac{3}{2} \cdot 24y^2 = \frac{3}{2} \cdot 6x = 9x$

$\Rightarrow x^4 = x \iff x(x^3 - 1) = 0 \Rightarrow x = 0$ or $x = 1$.

If $x=0$, we get $3 \cdot (0)^2 = 6y \Rightarrow y = 0$

If $x=1$, we get $3 \cdot (1)^2 = 6y \Rightarrow y = 1/2$

Now, using $H(f)$:

$Hf(0,0) = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix} \xrightarrow{\det} -36 < 0 \Rightarrow (0,0)$ is a saddle pt

$Hf(1, 1/2) = \begin{pmatrix} 6 & -6 \\ -6 & 24 \end{pmatrix} \xrightarrow{\det} 6 \cdot 24 - 36 = 6 \cdot (24 - 6) = 6 \cdot 18 > 0 \Rightarrow (1, 1/2)$ is a local min or max

since $f_{xx}(1, 1/2) > 0$, it is a local min

7. $\vec{r}(t) = (\cos(t), \sin(t), \cos(2t)) = (r_x(t), r_y(t), r_z(t))$

a) We have

$$(r_x(t))^2 + (r_y(t))^2 = \cos^2(t) + \sin^2(t) = 1 \Rightarrow \text{lies on cylinder } x^2+y^2=1$$

$$(r_x(t))^2 - (r_y(t))^2 = \cos^2(t) - \sin^2(t) = \cos(2t) = r_z \Rightarrow \text{lies on hyp. par. } z = x^2-y^2$$

b) Compute:

$$\vec{r}'(t) = (-\sin(t), \cos(t), -2\sin(2t)) \qquad \vec{r}'(0) = (0,1,0)$$

$$\Rightarrow$$

$$\vec{r}''(t) = (-\cos(t), -\sin(t), -4\cos(2t)) \qquad \vec{r}''(0) = (-1,0,-4)$$

$$\|\vec{r}'(0)\| = 1$$

notice that $\vec{r}'(0) \cdot \vec{r}''(0) = 0 \Rightarrow$ angle between $\vec{r}'(0)$ and $\vec{r}''(0) = \pi/2$

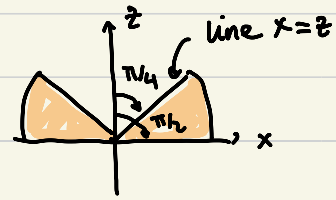
since $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin\theta$, we have $\|\vec{r}'(0) \times \vec{r}''(0)\| = \underbrace{\|\vec{r}'(0)\|}_{=1} \cdot \|\vec{r}''(0)\| \cdot \underbrace{\sin \pi/2}_{=1}$

Thus, $\|\vec{r}''(0)\| = \sqrt{(-1)^2 + 0^2 + (-4)^2} = \sqrt{17}$ and $\kappa(0) = \dfrac{\|\vec{r}'(0) \times \vec{r}''(0)\|}{\|\vec{r}'(0)\|^3} = \dfrac{\sqrt{17}}{1^3} = \sqrt{17}$.

8. Use spherical coordinates:

inside sphere $x^2+y^2+z^2=4 \Rightarrow \rho \in [0,2]$.

below half-cone and above xy plane $\Rightarrow \phi \in [\pi/4, \pi/2]$



Thus, we get

$$E = \{(\rho,\theta,\phi) : 0 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ \pi/4 \le \phi \le \pi/2\}$$

and

$$\text{volume} = \iiint_E dV = \int_0^2 \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \left(\int_0^2 \rho^2 d\rho\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_{\pi/4}^{\pi/2} \sin\phi \, d\phi\right)$$

$$= \left(\frac{8}{3}\right)\cdot(2\pi)\cdot\left(-\cos\phi\right)\Big|_{\pi/4}^{\pi/2} = \frac{8}{3}\cdot 2\pi \cdot \frac{\sqrt{2}}{2} = \frac{8\sqrt{2}}{3}\pi$$