final
math222, calculus III
summer 2025

Justify all your claims rigorously. This exam is worth 55% of the grade for math222 and contains 8 questions. Allotted time is 3 hours.
This is a closed book exam, no material (e.g. calculator, dictionary, crib sheet) is permitted except pen/pencil and eraser.

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useful formulas.

curves
Let $\vec{r} : (a, b) \to \mathbb{R}^3$ be a regular curve having no singular point of order 1. Then, the curvature κ and the torsion τ satisfy

$$\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} \qquad \tau(t) = \frac{(\vec{r}\,''(t) \times \vec{r}\,'(t)) \bullet \vec{r}\,'''(t)}{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|^2}$$

When $\vec{r}$ is parametrized by arc-length, the Frenet-Serret equations are

$$\vec{t}\,'(s) = \kappa(s)\vec{n}(s)$$
$$\vec{n}\,'(s) = -\kappa(s)\vec{t}(s) - \tau(s)\vec{b}(s)$$
$$\vec{b}\,'(s) = \tau(s)\vec{n}(s)$$

where $\vec{t}(s)$ is the tangent vector, $\vec{n}(s)$ is the normal vector and $\vec{b}(s)$ is the binormal vector.

coordinate systems

— The polar coordinates in $2d$ are given by

$$x(r, \theta) = r\cos(\theta) \qquad y(r, \theta) = r\sin(\theta)$$

where $r \in \mathbb{R}_{\geq 0}$ and $\theta \in [0, 2\pi)$.

— The spherical coordinates in $3d$ are given by

$$x(\rho, \theta, \phi) = \rho\cos(\theta)\sin(\phi) \qquad y(\rho, \theta, \phi) = \rho\sin(\theta)\sin(\phi) \qquad z(\rho, \theta, \phi) = \rho\cos(\phi)$$

where $\rho \in \mathbb{R}_{\geq 0}$, $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$.

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1. (6 points) Compute the Taylor series centred at zero of the function $f(x) = xe^{2x}$ and find its interval of convergence.

2. (6 points) Evaluate the following integrals :

a. $\displaystyle\int_0^{\sqrt{2\pi}} \int_{y/2}^{\sqrt{2\pi}/2} \cos(x^2)\, \mathrm{d}x\mathrm{d}y$

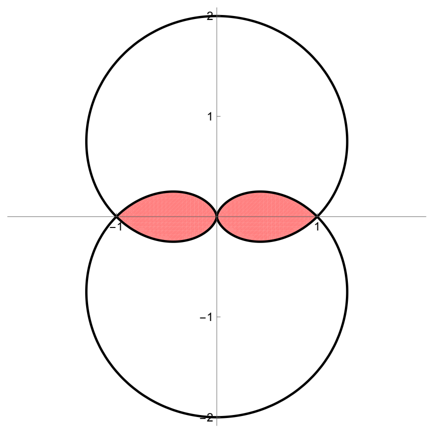
b. $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{1} xy\, \mathrm{d}z\mathrm{d}x\mathrm{d}y$

3. (8 points) Find the minima and maxima of the function $f(x, y) = e^{-xy}$ subject to the constraint $x^2+4y^2 = 1$.

4. (8 points) Show that if $z = f(x, y)$ and $x = r\cos(\theta)$, $y = r\sin(\theta)$, then

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{r^2} \cdot \frac{\partial^2 z}{\partial \theta^2} + \frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial z}{\partial r}.$$

5. (8 points) Compute the area of the shaded region



where the two curves are given in polar coordinates by the equations $r = 1 + \sin(\theta)$ and $r = 1 - \sin(\theta)$.

6. (8 points) Find the critical points of the function

$$f(x, y) = x^3 - 6xy + 8y^3$$

and determine if the critical points are minima, maxima or saddle points.

7. (8 points) Consider the curve $\vec{r} : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\vec{r}(t) = (\cos(t), \sin(t), \cos(2t)).$$

a. Show that the trace of $\vec{r}$ lies on the cylinder $x^2 + y^2 = 1$ and on the hyperbolic paraboloid $z = x^2 - y^2$.

b. Show that the curvature of $\vec{r}$ at $t = 0$ is $\sqrt{17}$.

8. (8 points) Compute the volume of the solid lying above the xy plane, inside the sphere $x^2 + y^2 + z^2 = 4$ and below the half-cone $z = \sqrt{x^2 + y^2}$.