MATH133 – Summer 2022 – Tutorial 6

All questions denoted $(*)$ are not crucial to the course, but they give a taste of further topics in mathematics. I strongly encourage you to attempt all problems in this tutorial, but do not worry if you get stuck!

1. *Shortest distances.* Let $a, b, u \in \mathbb{R}^3$, and assume that a is not in the line L generated by $b + t \cdot u$. Let x be the point on L such that the distance between a and x is the shortest distance from a to L. Show that u is perpendicular to $x - a$. Conclude the same for the shortest distance from a point to a plane.

2. *Gaussian Elimination.* Find solutions, if they exist, for the following systems of equations using Gaussian Elimination

   (a) $\begin{cases} x + y = 0 \\ x - y = 0 \end{cases}$

   (b) $\begin{cases} x + y - 2z + w = 0 \\ x - y - w = 0 \\ 2y - z + 3w = 0 \end{cases}$

   (c) $\begin{cases} x + y - 2z + w = 1 \\ x - y - w = 5 \\ 2y - z + 3w = 3 \end{cases}$

   (d) $\begin{cases} x + y - 2z = 0 \\ x - y = 0 \\ 2y - 2z = 0 \end{cases}$

3. *Idempotency.* An $\mathbb{F}$-linear map $f : V \to V$ is called idempotent if $f \circ f = f^2 = f$. Show that for any non-zero $v \in \mathbb{R}^n$, proj_v is idempotent.

4. *Nilpotency.* An $\mathbb{F}$-linear map $f : V \to V$ is called nilpotent if there exists an $n \in \mathbb{N}$ such that $f^n = 0$. Show that any $n \times n$ matrix of the form

$$\begin{bmatrix} 0 & x_{12} & \dots & x_{1n} \\ 0 & 0 & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

   is nilpotent. *Note: such a matrix is called strictly upper triangular.*