MATH133 – Summer 2022 – Tutorial 1

All questions denoted $(*)$ are not crucial to the course, but they give a taste of further topics in mathematics. I strongly encourage you to attempt all problems in this tutorial, but do not worry if you get stuck!

1. *Natural numbers.* What is the difference between the following two statements? Which ones are true?
   (a) $\forall x \in \mathbb{N}(x + 1 \in \mathbb{N})$.
   (b) $\forall x \in \mathbb{N} \exists y \in \mathbb{N}(x = y + 1)$.
2. *True or False.* Which of the following statements are true?
   (a) $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z}(x + y = 0)$.
   (b) $\forall x \in \mathbb{N} \exists y \in \mathbb{N}(x + y = 0)$.
   (c) $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z}(x \cdot y = 0)$.
   (d) $\forall x \in \mathbb{Q} \exists y \in \mathbb{Q}(x \cdot y = 1)$.
   (e) $\forall x \in \mathbb{R} \exists y \in \mathbb{Q}(x + y = 0)$.
   (f) $\exists x \in \mathbb{N} \forall y \in \mathbb{N}(x \cdot y = y)$.
   (g) $\exists x \in \mathbb{Z} \forall y \in \mathbb{Z}(x \cdot y = 0)$.
   (h) $\exists x \in \mathbb{R} \forall y \in \mathbb{R}(x + y = 1)$.
3. *Membership and subsets.* Which of the following is true? Which one is false?
   (a) $\emptyset \in \Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\Big\}$
   (b) $\emptyset \subseteq \Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\Big\}$
   (c) $\{\emptyset\} \in \Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\Big\}$
   (d) $\{\emptyset, \{\emptyset\}\} \in \Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\Big\}$
   (e) $\{\emptyset, \{\emptyset\}\} \subseteq \Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\Big\}$
   (f) $\{\{\emptyset\}\} \in \Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\Big\}$
4. *Distributivity.* Prove the following:
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

   These are called the distributivity laws for sets, and are very similar to the distributivity laws for addition and multiplication.

5. *DeMorgan.* Prove the following:
   (a) $(A \cup B)^c = A^c \cap B^c$.
   (b) $(A \cap B)^c = A^c \cup B^c$.

6. *Functions.* Let $f(x) = x^2$, $g(x) = cos(x) + e^x$, $h(x) = \frac{1}{x}$. Find the following, if they exist.
   (a) $f([-1, 1])$.
   (b) $f^{-1}([0, 1])$.
   (c) $f^{-1}(\{1\})$.
   (d) $g([0, 1])$.
   (e) $g^{-1}([-1, 1])$.
   (f) $h(0)$.
   (g) $h((0, 1])$.
   (h) $h^{-1}(\{0\})$.

7. *Preimages.* Let $f : A \to B$ and $A' \subseteq A, B' \subseteq B$. Prove the following:
   (a) $A' \subseteq f^{-1}(f(A'))$.
   (b) $f(f^{-1}(B')) \subseteq B'$.

8. *-jections.* Try to find some injective, surjective, and bijective functions:
   (a) from $[0, 1]$ to $[0, 1]$.
   (b) from $(0, 1)$ to $(1, 2)$.
   (c) from $[0, \pi]$ to $(2, 3]$.
   (d) from $\{0, 1, 2\}$ to $\{0, 1\}$.
   (e) $(*)$from $\mathbb{N}$ to $\mathbb{Z}$.
   (f) $(*)$from $\mathbb{N}$ to $\mathbb{N} \cup \left\{\frac{1}{2}\right\}$.

   The solution to the last question is called Hilbert's hotel, a fundamental result in set theory.

9. $(*)$*Ordered pairs.* An ordered pair is defined $(x, y) := \{\{x\}, \{x, y\}\}$. Show that $(x_0, y_0) = (x_1, y_1)$ if and only if $x_0 = x_1$ and $y_0 = y_1$. *Remember that sets are unordered!*