Exponentially many perfect matchings in cubic graphs

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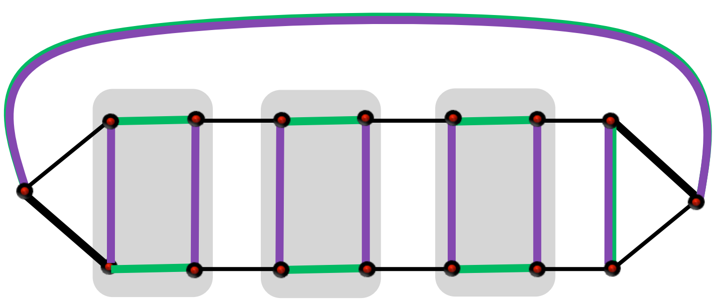
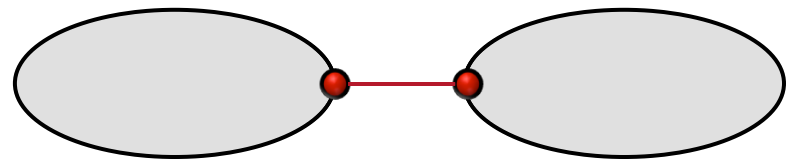


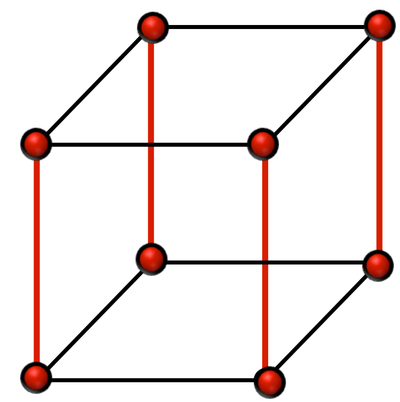
Perfect matchings in bridgeless cubic graphs

A bridge : an edge whose deletion disconnects the graph



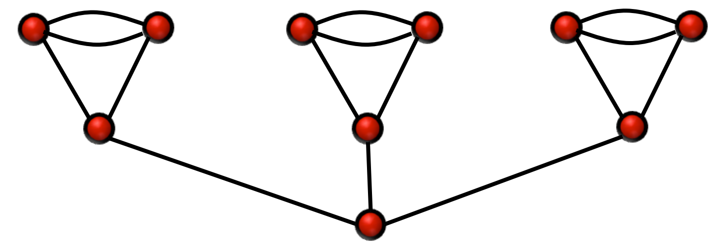
Cubic graph: Every vertex is incident to exactly 3 edges

Perfect matching: A set of edges that covers all vertices exactly once.



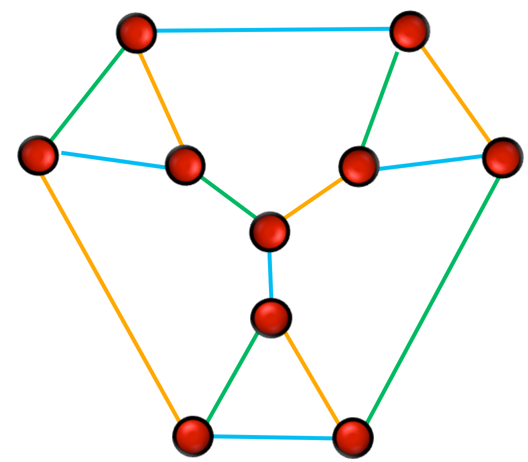
Perfect matchings in bridgeless cubic graphs

Theorem(Petersen, 1891): Every bridgeless cubic graph has a perfect matching.



Observation(Tait, 1880): The Four Color Theorem is equivalent to the following:

The edge set of every planar cubic bridgeless graph is the union of three perfect matchings.

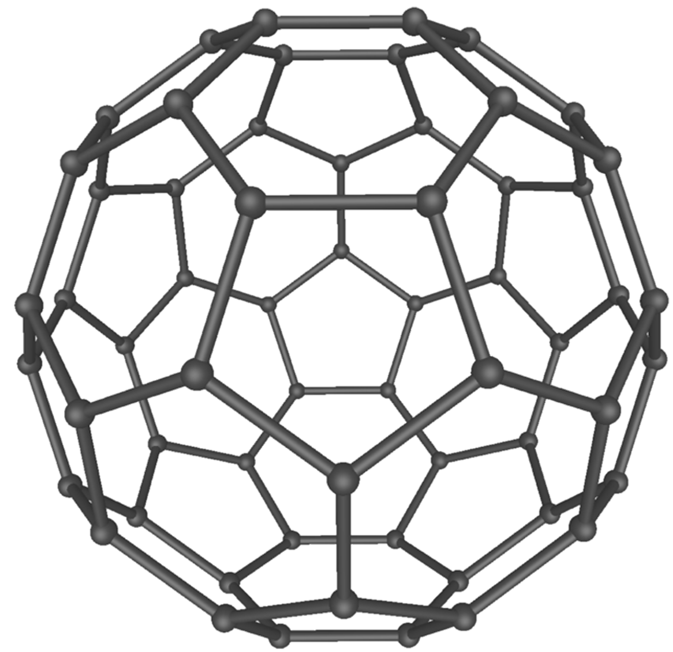


Conjecture(Berge, Fulkerson, 1971):
In every bridgeless cubic graph there exists a collection of perfect matchings covering every edge exactly twice.

The number of perfect matchings

$m(G)$: The number of perfect matchings in a graph G

- $m(G)$ is hard to compute (Valiant, 1979)
- $m(G)$ is equal to the permanent of the graph biadjacency matrix when G is bipartite
- $m(G)$ is related to meaningful chemical and physical properties of molecules represented by G



Perfect matchings in bridgeless cubic graphs

$m(G)$: The number of perfect matchings in a graph G

Theorem: There exists a constant $\varepsilon > 0$ such that $m(G) \geq 2^{\varepsilon |V(G)|}$ in every cubic bridgeless graph G. ($\varepsilon = 1/3656.$)

Conjectured by Lovász and Plummer (1970's).

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Previous results:

Voorhoeve (1979) : $m(G) \geq \left(\frac{4}{3}\right)^{|V(G)|/2}$ for bipartite G.

Chudnovsky, Seymour (2008): $m(G) \geq 2^{\varepsilon|V(G)|}$ for planar G. ($\varepsilon = 1/655978752.$)

Edmonds, Lovász, Pulleyblank(1982): $m(G) \geq n/4+2$ ($|V(G)| = n$)

Král', Sereni, Stiebitz (2008): $m(G) \geq n/2$

Esperet, Král', Škoda, Škrekovski (2009): $m(G) \geq 3n/4-10$

Esperet, Kardoš, Král' (2010): $m(G)$ is superlinear.

$m^*(G)$

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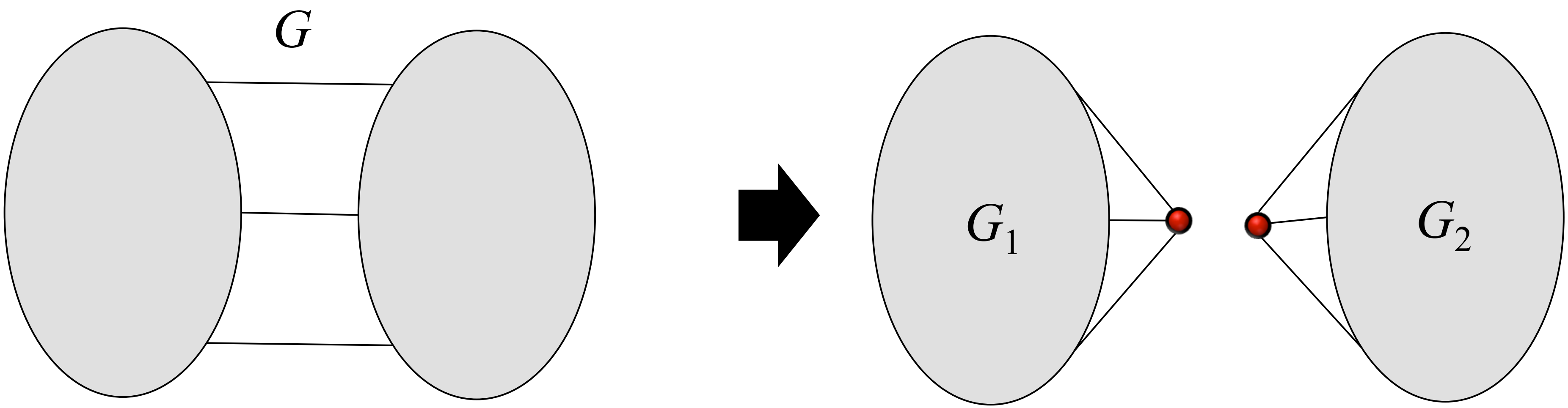


$$m^*(G) \geq m^*(G_1)\, m^*(G_2),$$

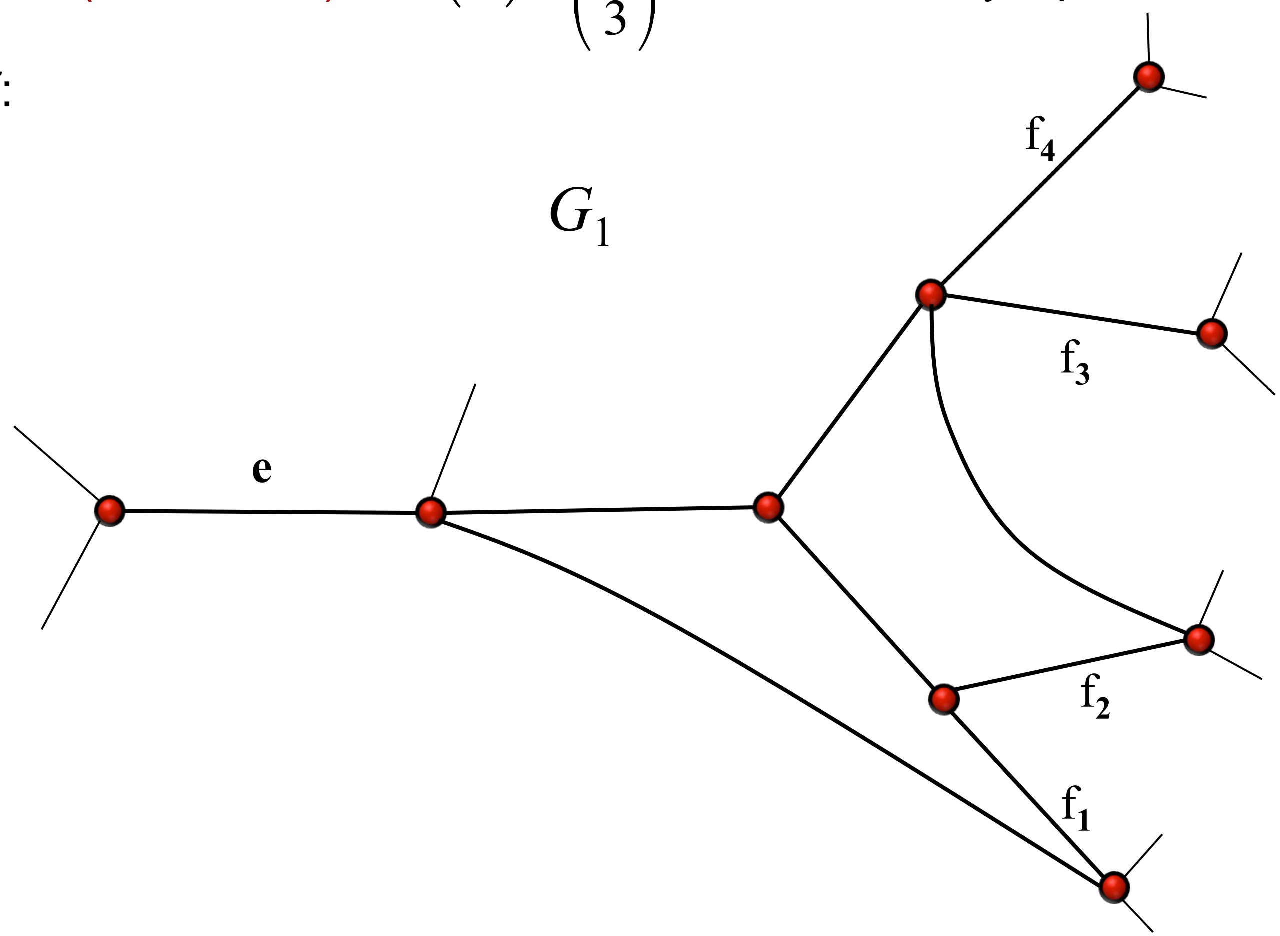
We can not say the same for $m(G)$.

Voorhoeve's splitting trick

$m^*(G)$: the maximum number k such that every edge of G belongs to at least k perfect matchings.

Theorem(Voorhoeve): $m^*(G) \geq \left(\frac{4}{3}\right)^{|V(G)|/2-3}$ for every bipartite cubic graph G.

Proof:



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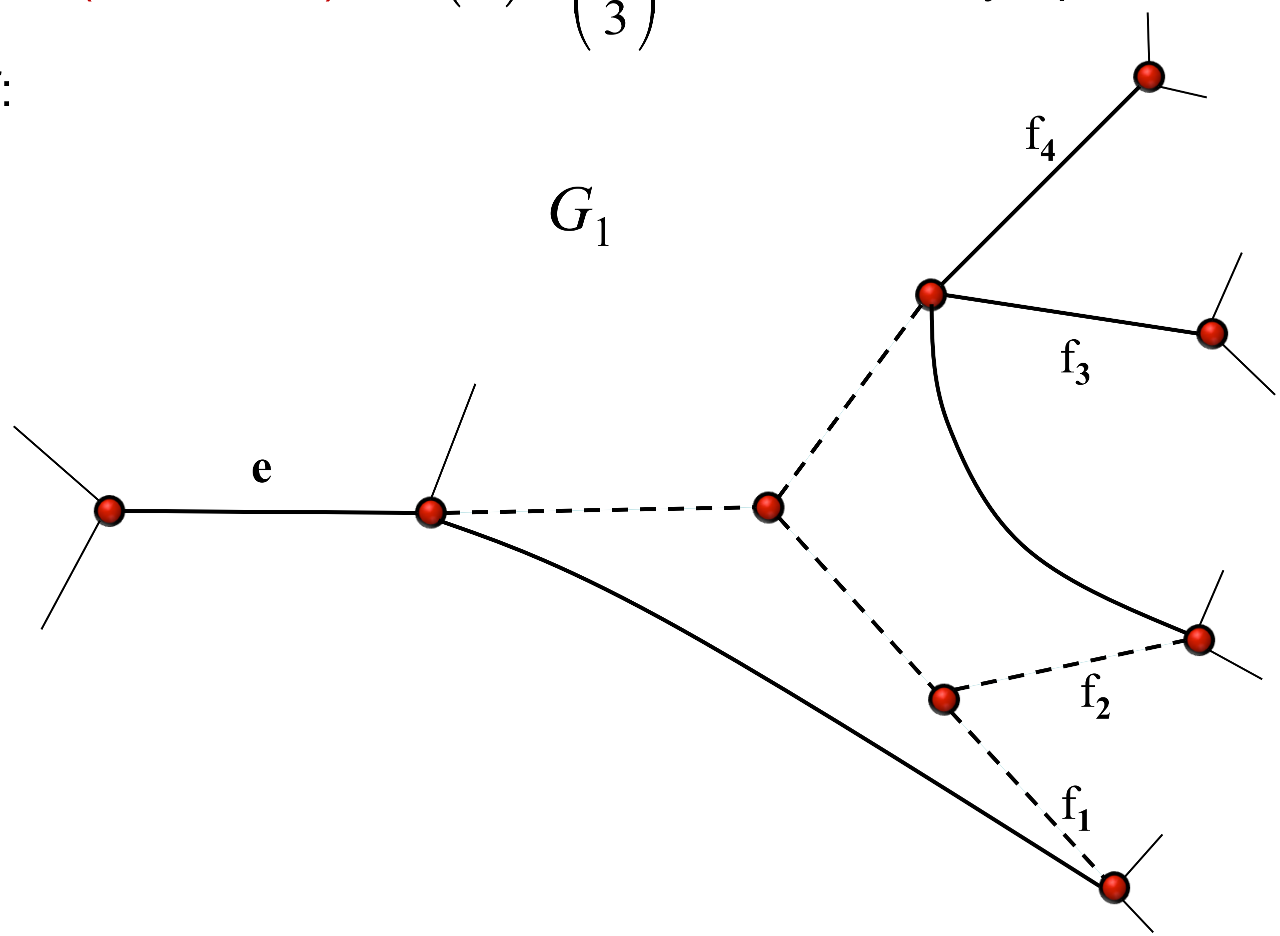

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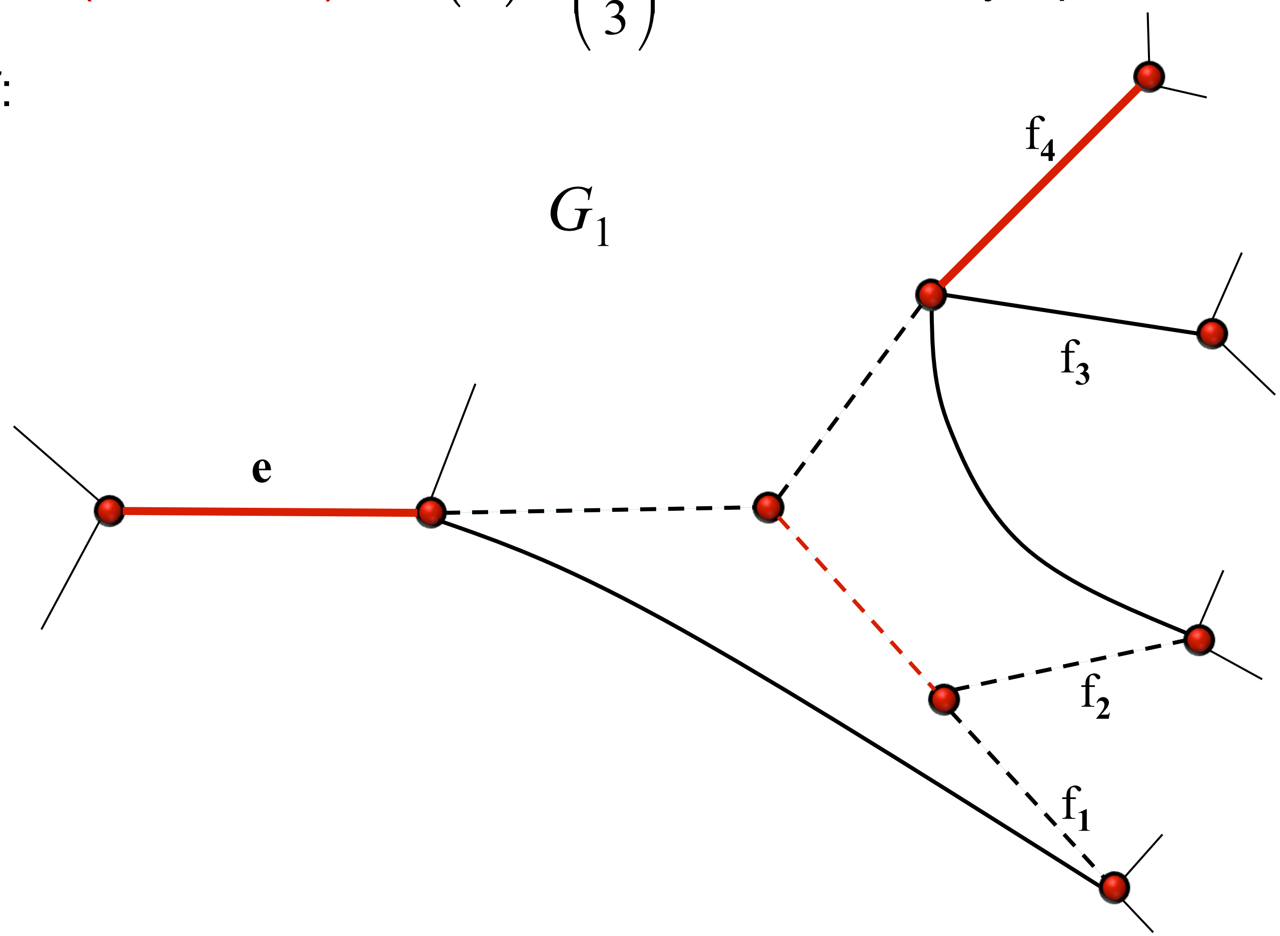

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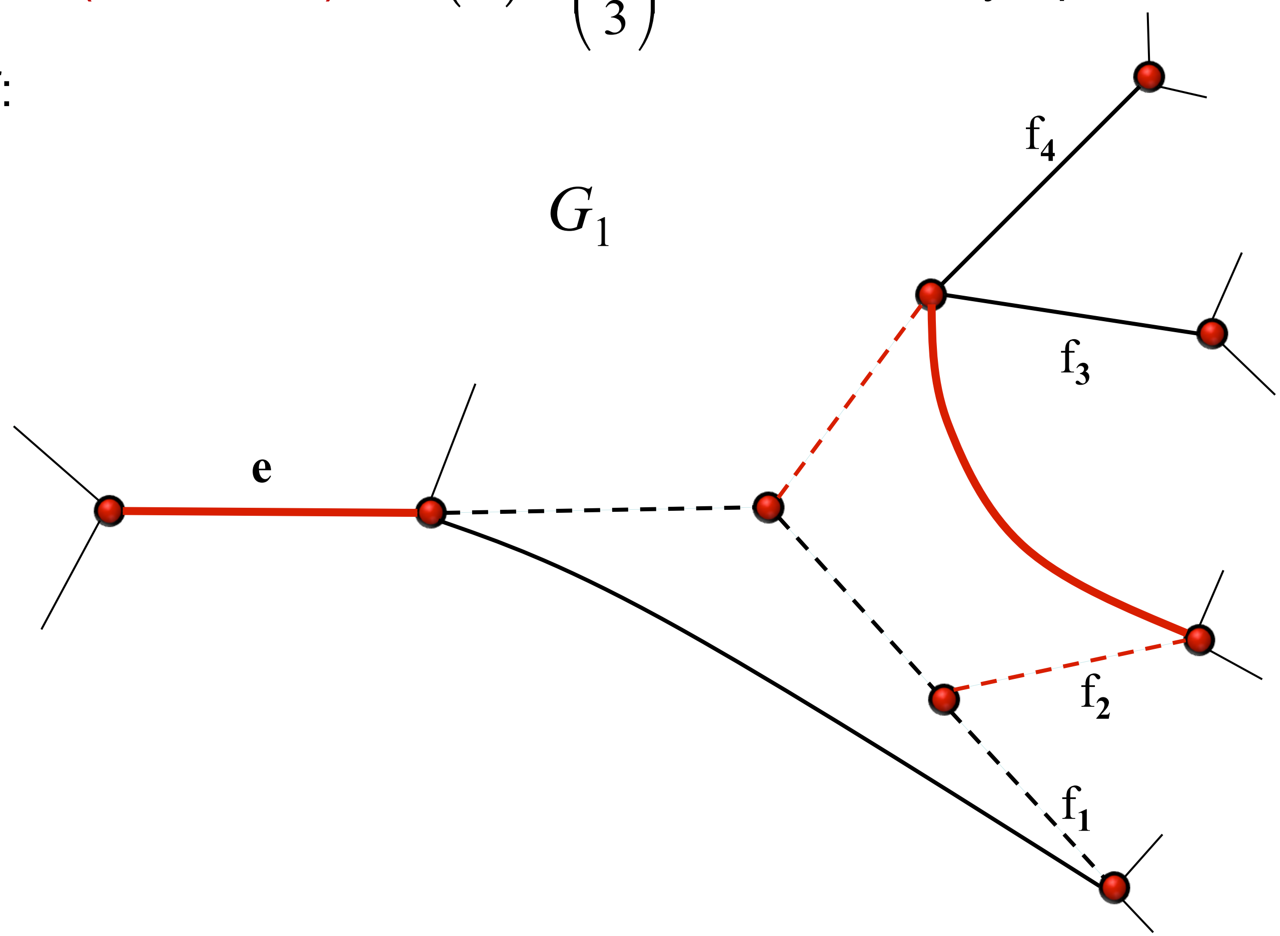

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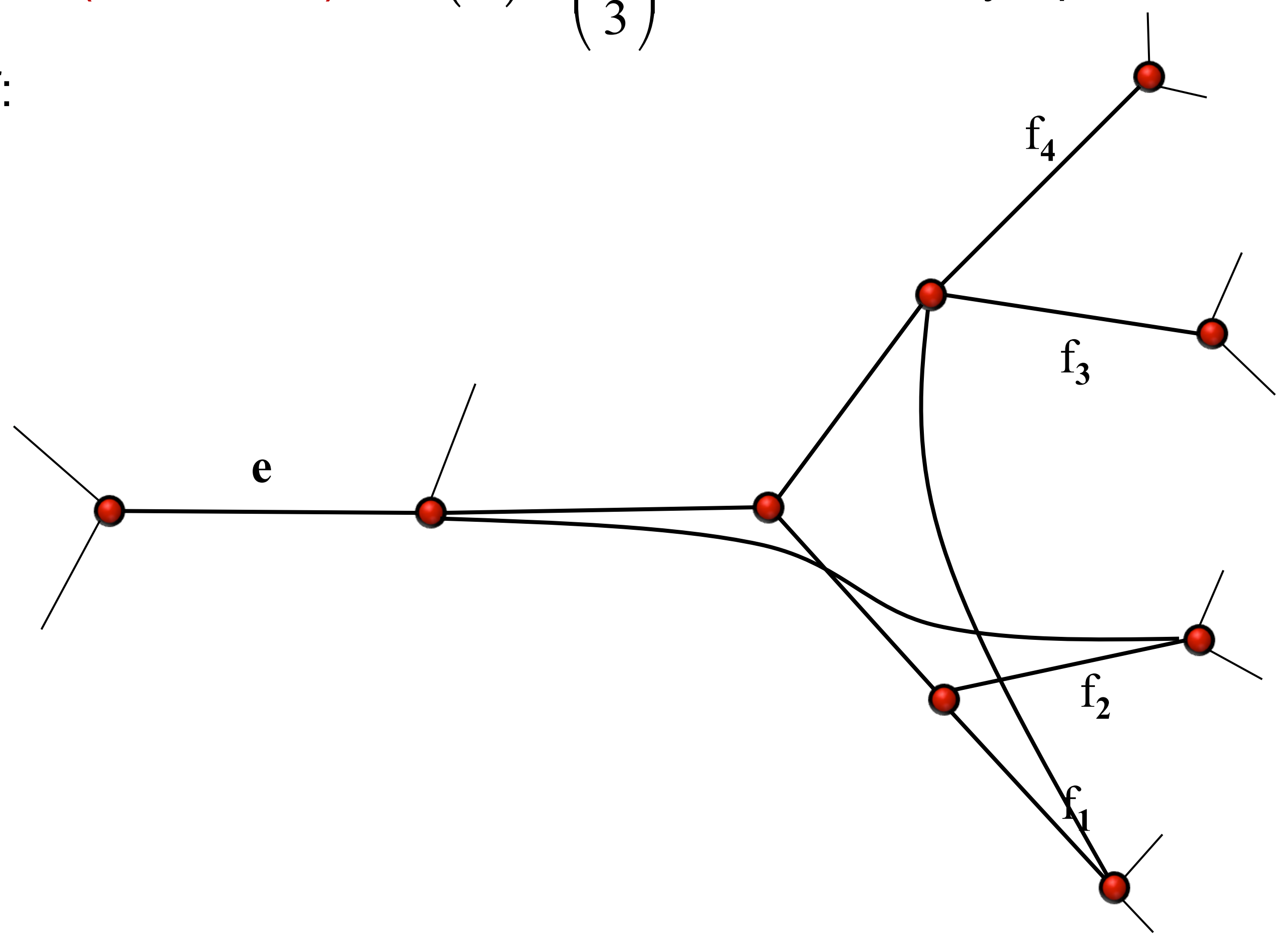

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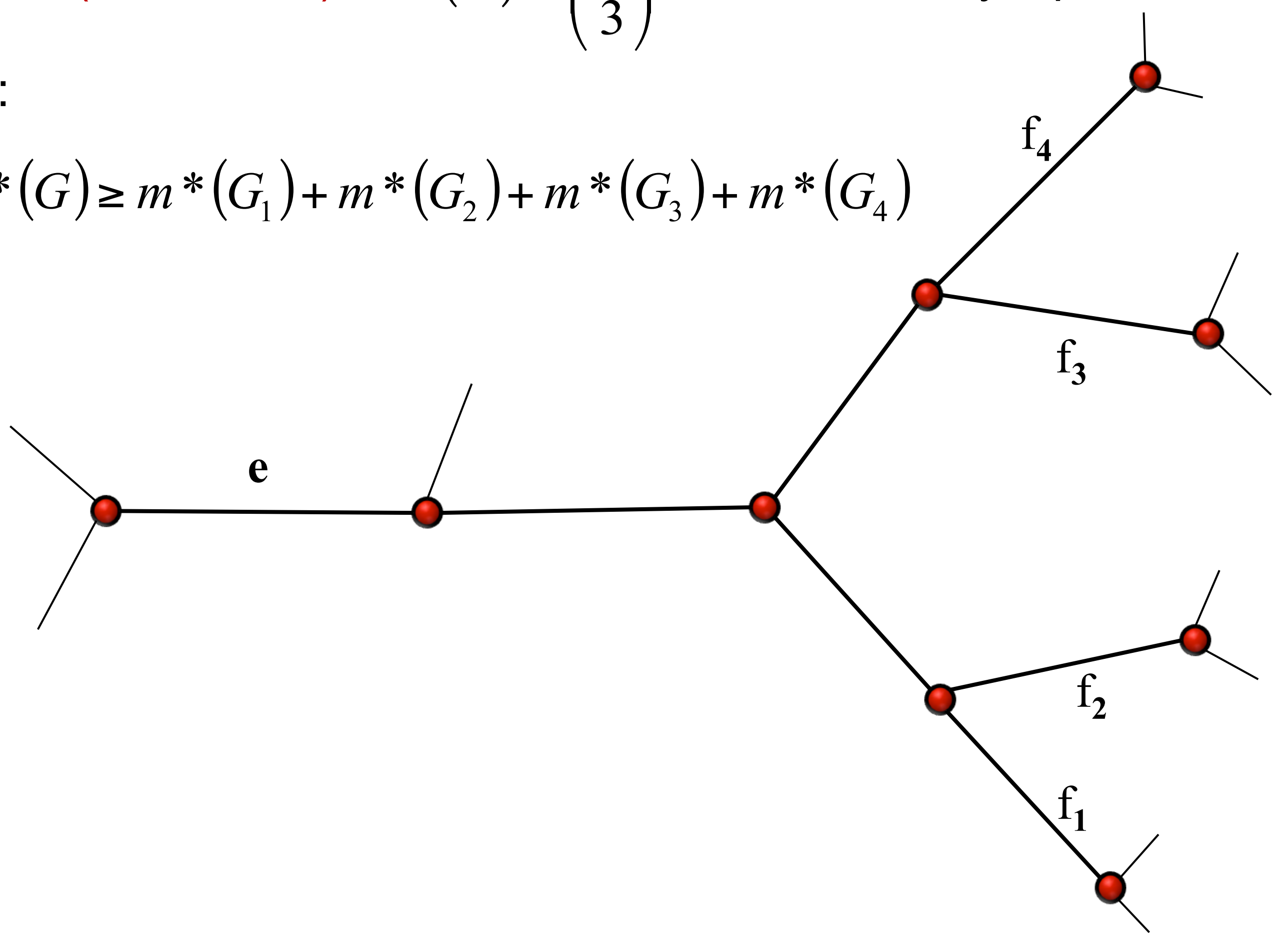
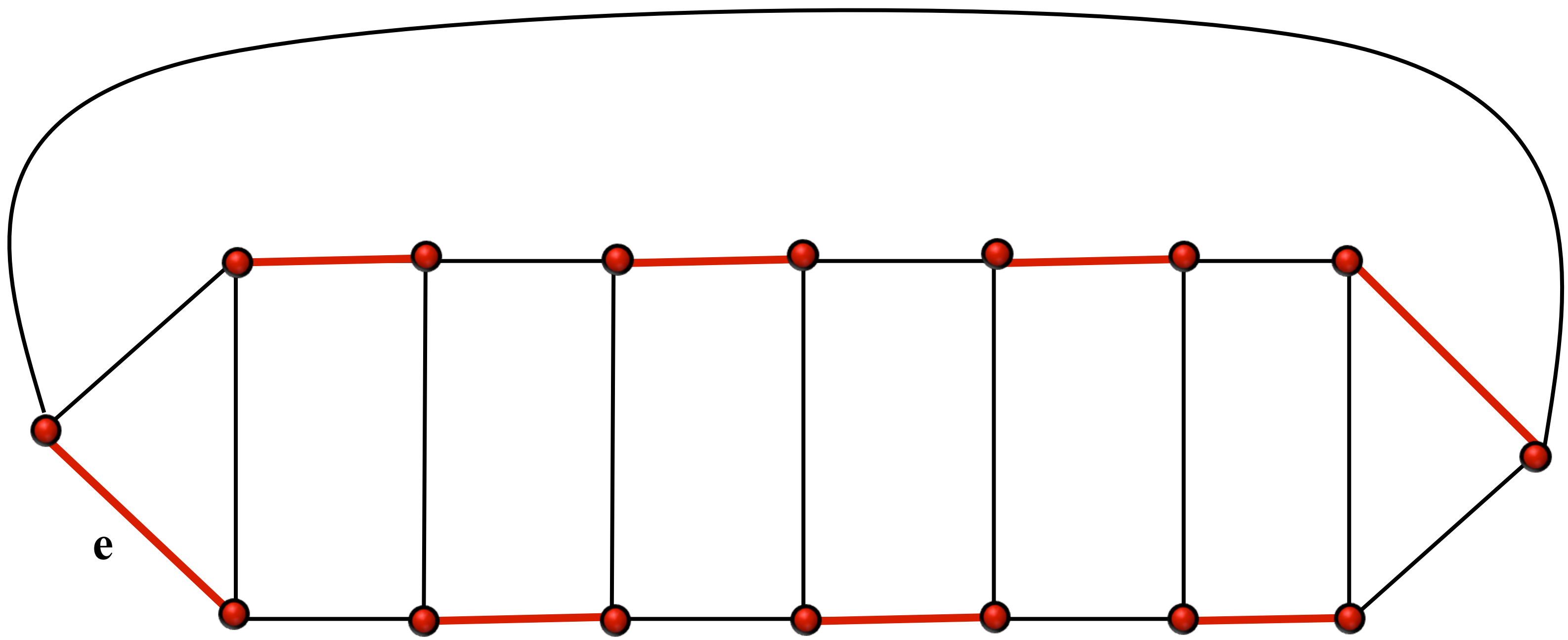
Proof:

$$3m^*(G) \geq m^*(G_1) + m^*(G_2) + m^*(G_3) + m^*(G_4)$$



Cubic bridgeless graph with $m^*(G)=1$



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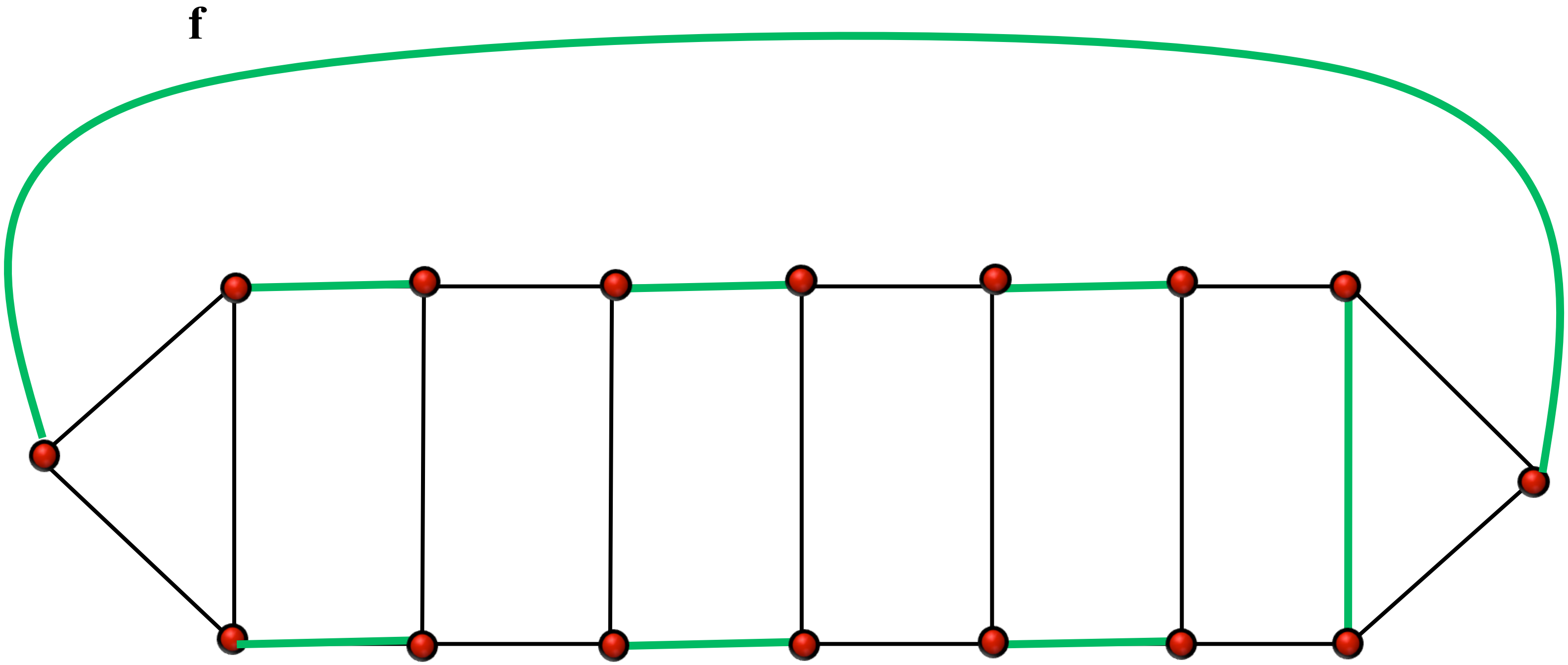

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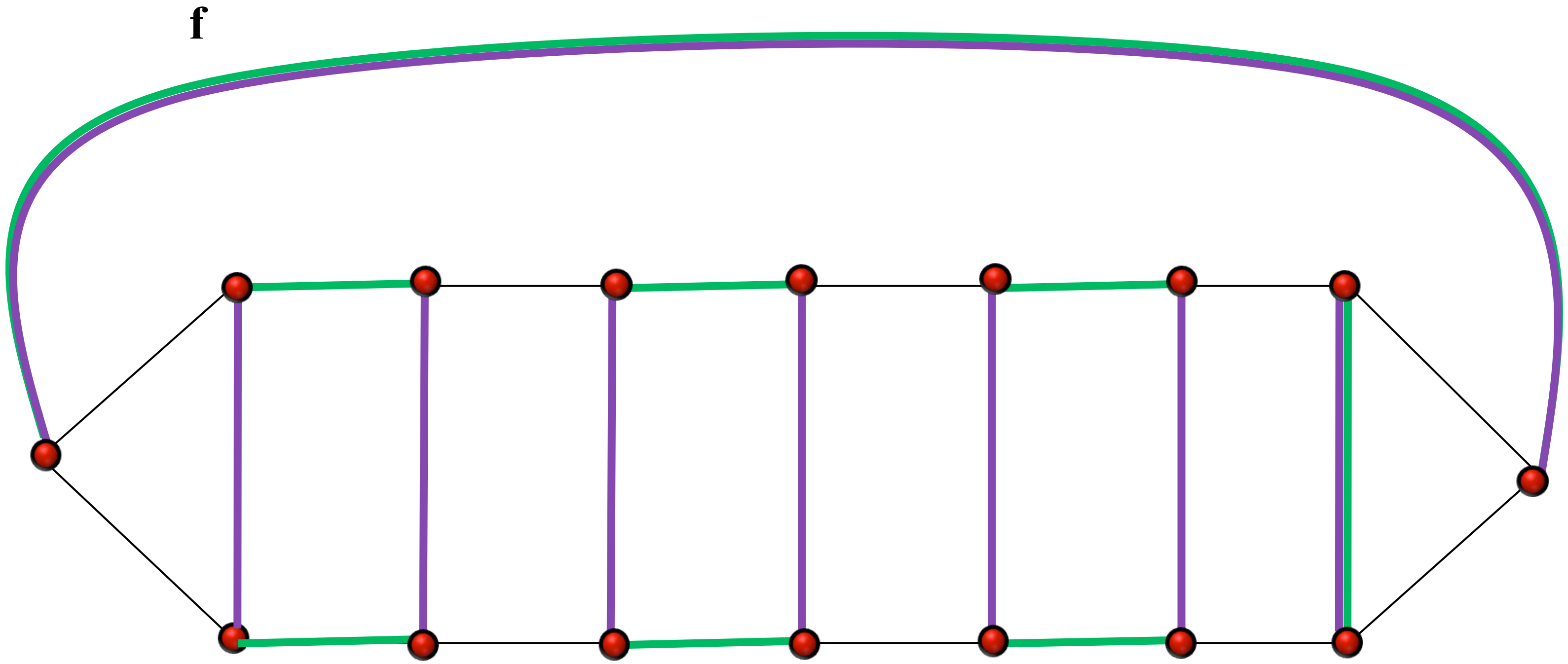

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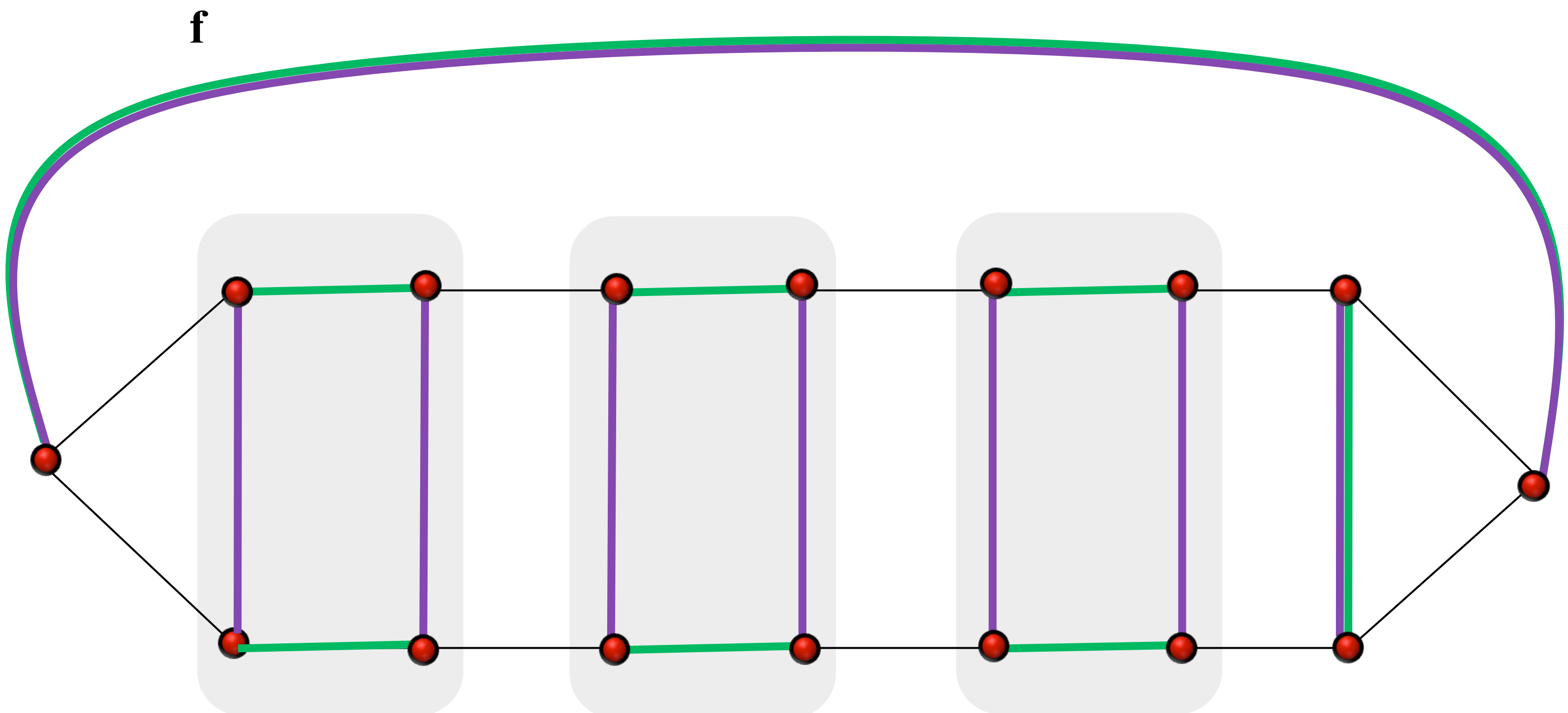

Two perfect matchings M_1 and M_2 such that M_1 [M_2 contains at least $\varepsilon|V(G)|$ disjoint cycles.

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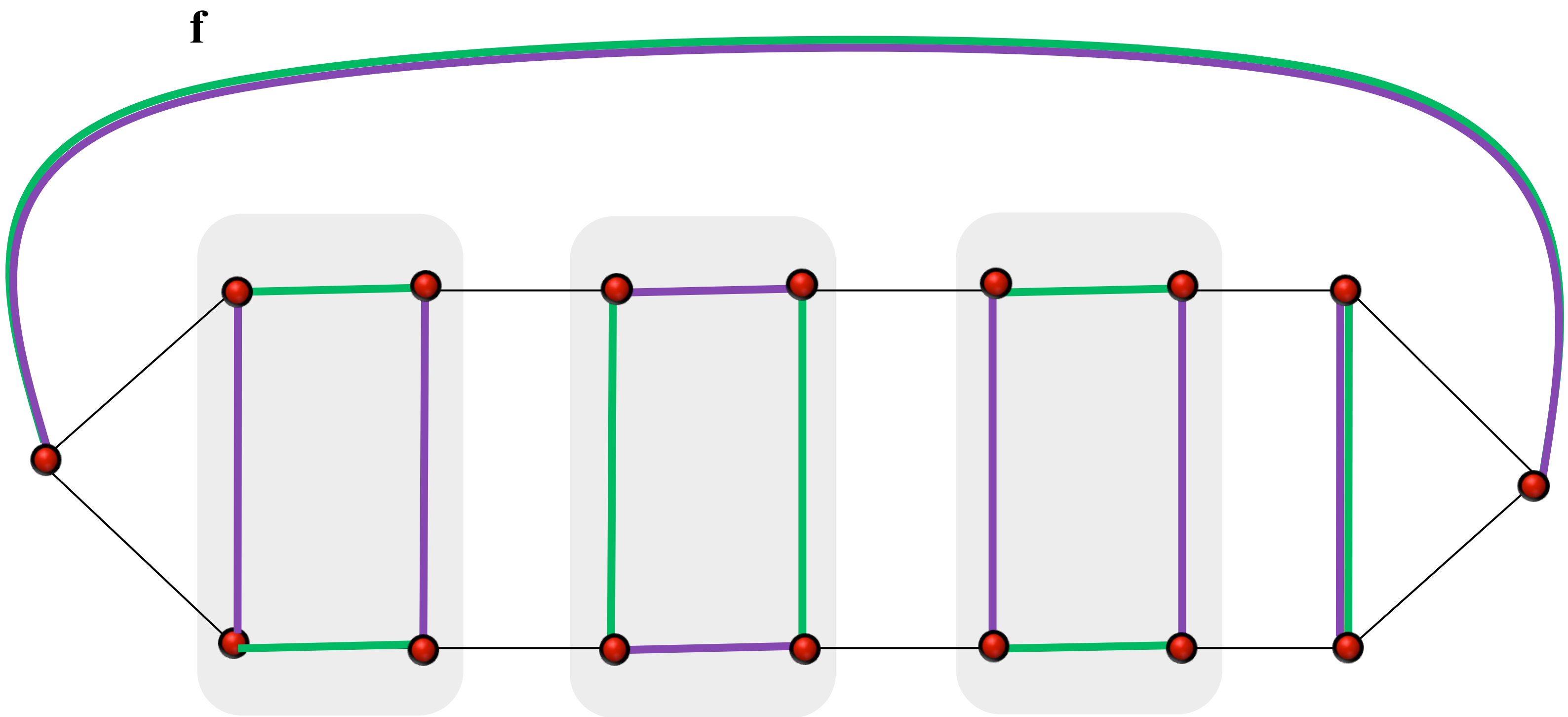


Two perfect matchings M_1 and M_2 such that $M_1 \cup M_2$ contains at least $\varepsilon|V(G)|$ disjoint cycles.

A strengthening

Theorem: There exists a constant $\varepsilon > 0$ such that for every cubic bridgeless graph G either

- $m^*(G) \geq 2^{\varepsilon|V(G)|}$ or
- for some two perfect matchings M_1 and M_2 in G the edge set M_1 [M_2 contains at least $\varepsilon|V(G)|$ disjoint cycles.

The perfect matching polytope

With a perfect matching M we associate a vector $\chi_M \in R^{E(G)} : \chi_M(e) = \begin{cases} 1, & e \in M \\ 0, & e \notin M \end{cases}$

The perfect matching polytope *PMP(G)* is the convex hull of characteristic vectors of perfect matchings of G.

Let $\delta(X)$ denote the set of edges in the cut separating X from $V(G)$-X.

Theorem(Edmonds): We have $w \in PMP(G)$ if and only if

- $0 \le w(e) \le 1$ for every $e \in E(G)$,
- $w(\delta(v)) = 1$ for every $v \in V(G)$,
- $w(\delta(X)) \ge 1$ for every odd $X \subseteq V(G)$.

A vector $w \in PMP(G)$ corresponds to a probabilistic distribution on the set of perfect matchings of G such that

$$\Pr[e \in M_w] = w(e).$$

If G is cubic and bridgeless then $w \equiv 1/3 \in PMP(G)$.

Burls

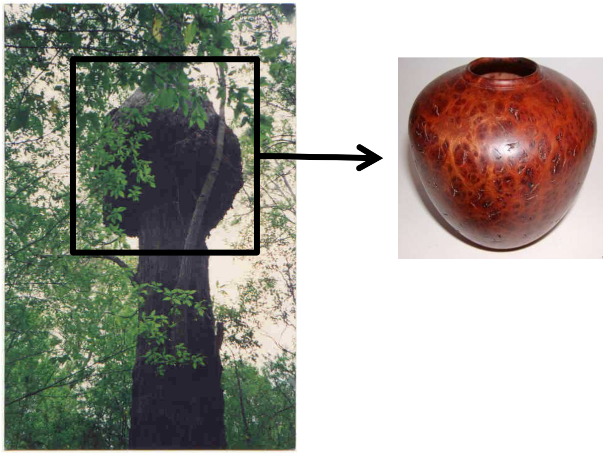
A set $X \subseteq V(G)$ is M-alternating for a perfect matching M of G if there exists another perfect matching M' such that M only differs from M' on X.

A set $X \subseteq V(G)$ is a burl if for every probabilistic distribution M_w such that

$$\Pr[e \in M_w] = 1/3,$$

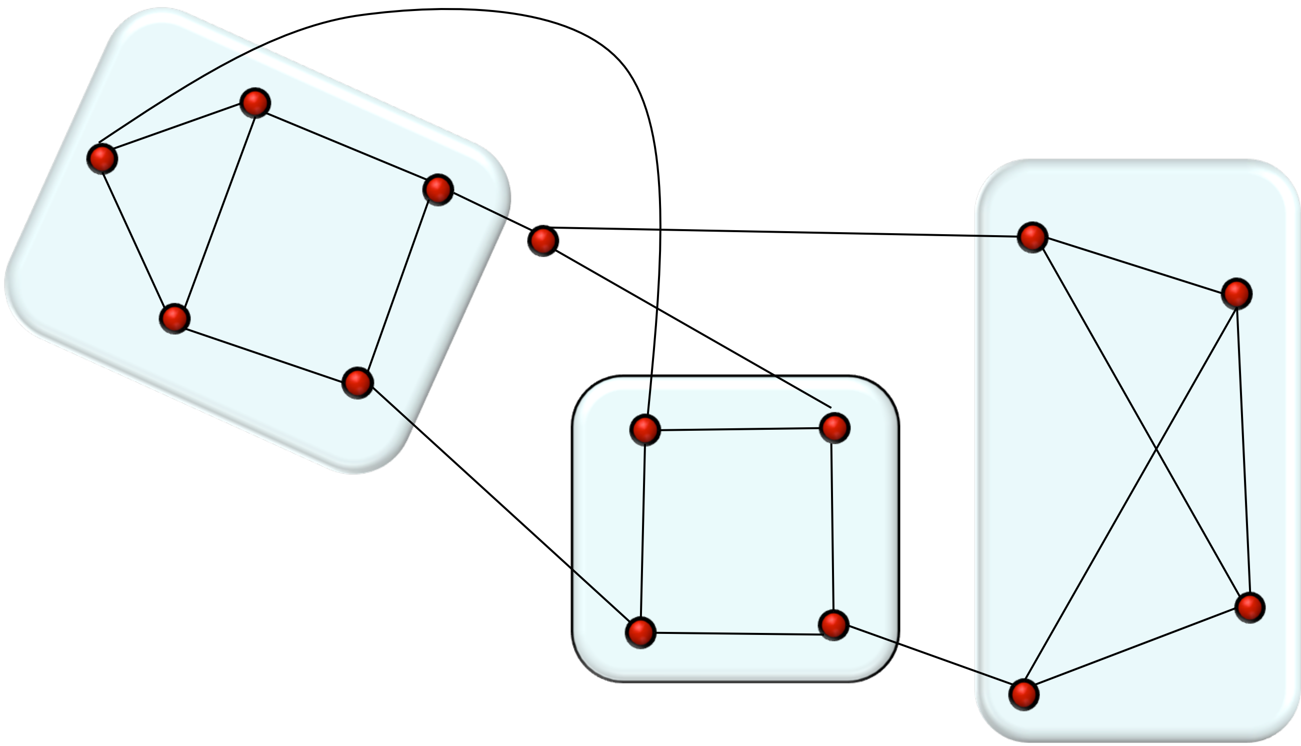
we have

$$\Pr[X \text{ is } M_w\text{-alternating}] \geq 1/3.$$



A foliage in G is a collection of pairwise disjoint burls.

A foliage



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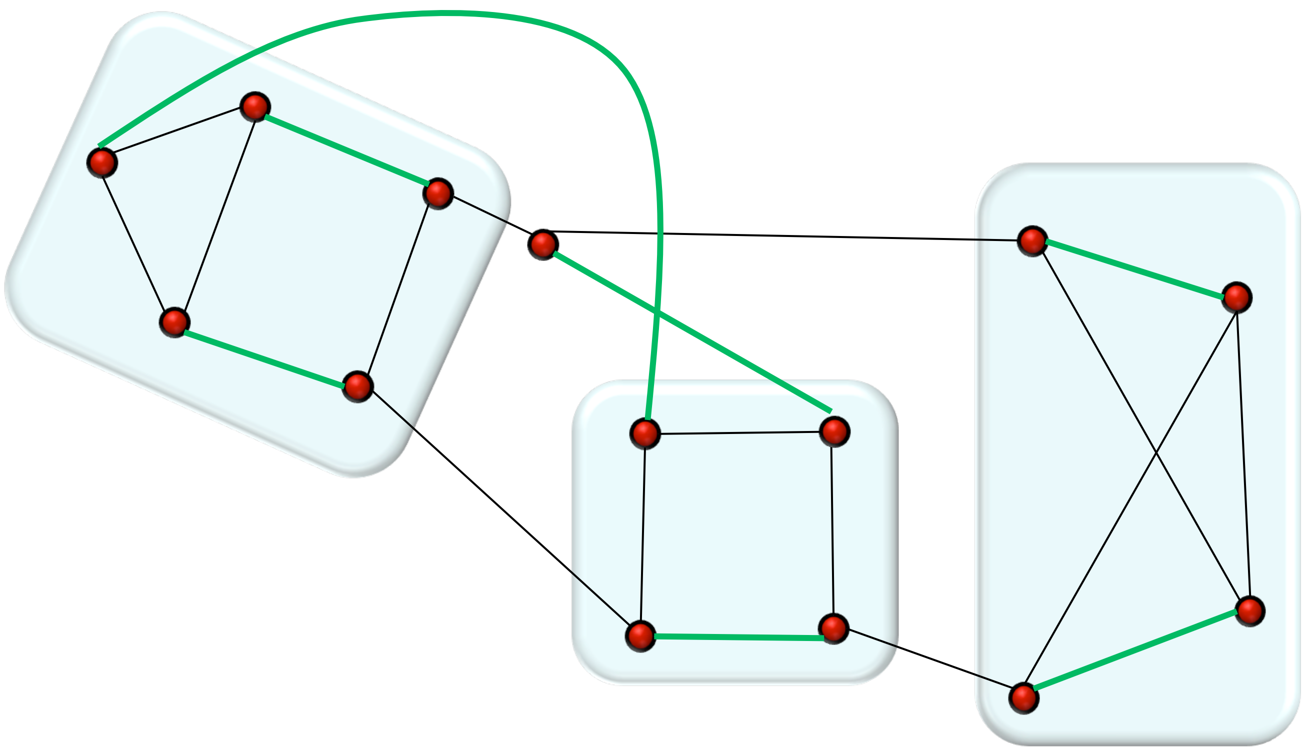

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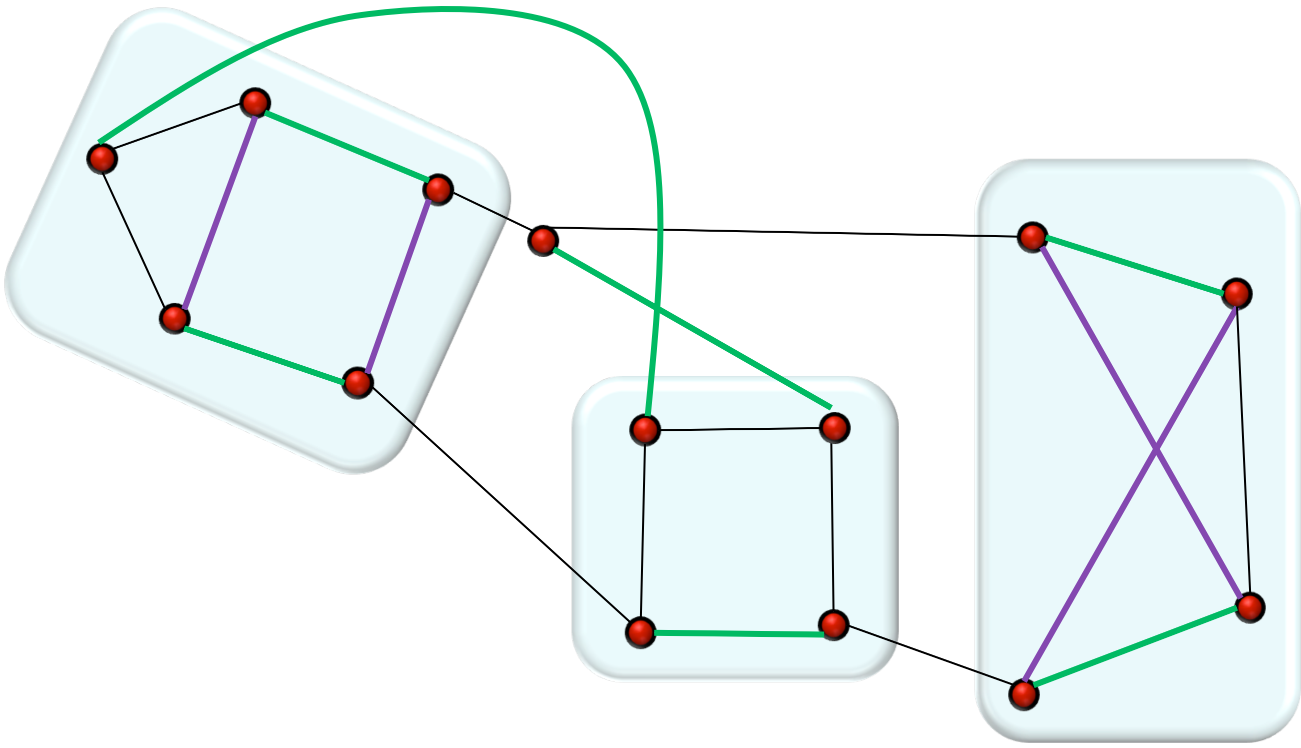

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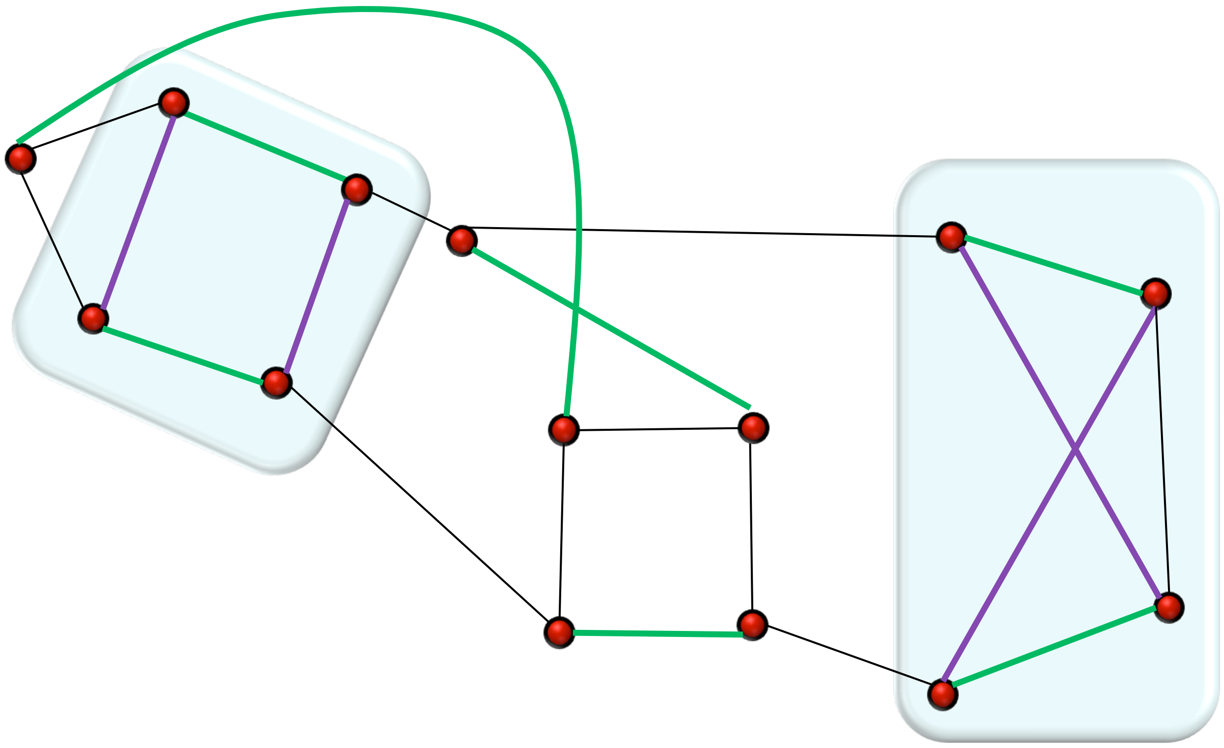

Using a foliage

A foliage in G is a collection of pairwise disjoint burls.
Let $f(G)$ denote the maximum size of a foliage in G.

Lemma: $m(G) \geq 2^{f(G)/3}$

Proof: Given a foliage $\{X_1, X_2, \ldots, X_k\}$ there exists $w \in PMP(G)$ such that each X_i is w-alternating. Then

$$\Pr[X_i \text{ is } M_w\text{-alternating}] \geq 1/3.$$

$$\mathrm{E}[|\{i : X_i \text{ is } M_w\text{-alternating}\}|] \geq k/3.$$

A perfect matching achieving the expected value can be independently changed to another perfect matching on each of the $k/3$ disjoint burls.

Examples of burls: Twigs

Lemma: For a cubic bridgeless graph *G*,

- $m^*(G) \geq 1$,
- $m^*(G) \geq 2$, if $|V(G)| \geq 6$ and G has no non-trivial cuts of size ≤ 3,
- $m(G) \geq 4$, if $|V(G)| \geq 6$.

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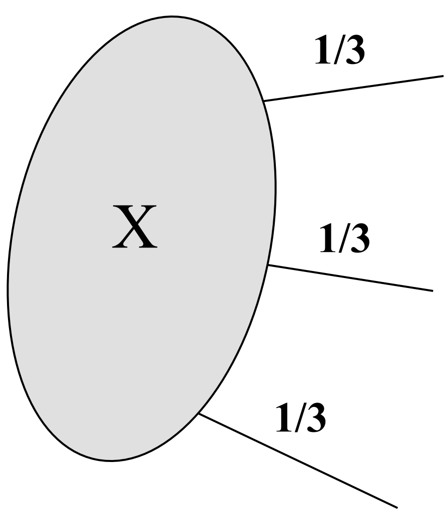
A set $X \subseteq V(G)$ is a twig if either $|\delta(X)| = 2$, or $|\delta(X)| = 3$ and $|X| \leq 5$.

Lemma: Every twig is a burl.

Proof: $\Pr[|M_w \cap \delta(X)| = 1] = 1.$



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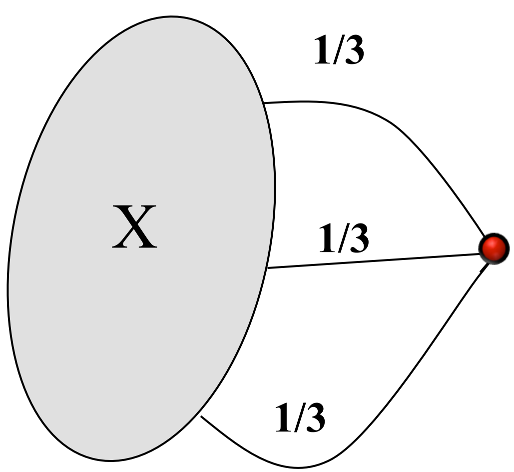
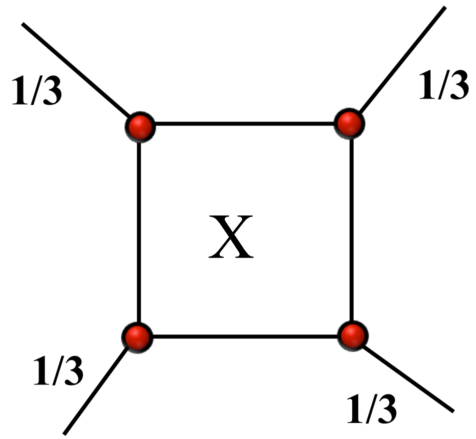


One of the edges of $\delta(X)$ is in at least 2 perfect matchings of the new graph.

Examples of burls: 4-cycles



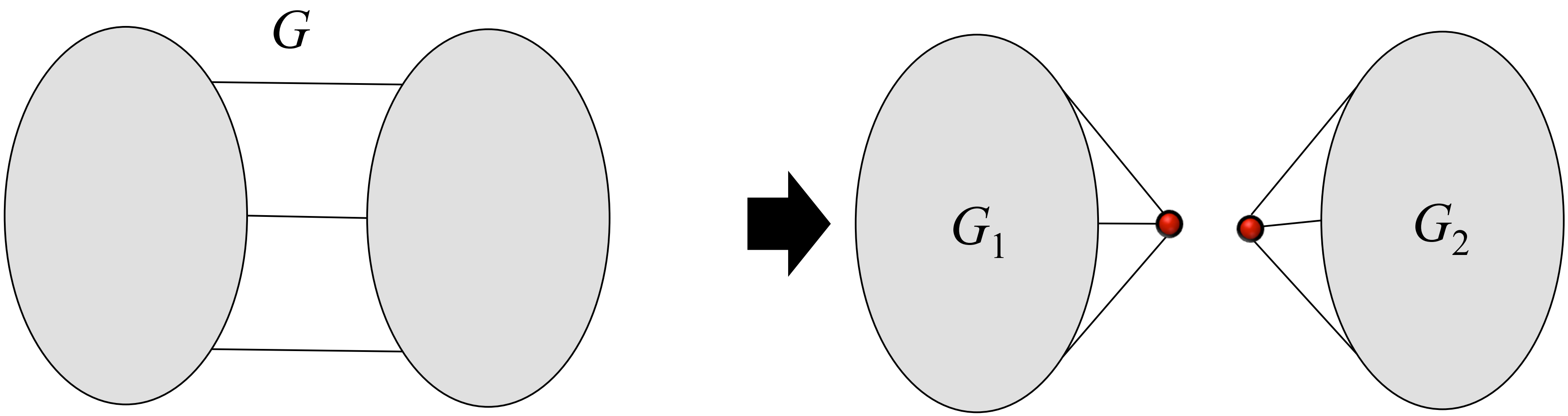
Lemma: Vertex set of any cycle of length 4 is a burl.

Proof: $$\mathrm{E}[|M_w \cap \delta(X)|] = 4/3,$$

$$|M_w \cap \delta(X)| \in \{0, 2, 4\},$$

$$\Pr[|M_w \cap \delta(X)| = 0] \geq 1/3.$$

Decomposing along small cuts



We say that G_1 and G_2 are obtained from G by a cut contraction.
(We can apply a similar procedure to cuts of size 2.)

Lemma: $m^*(G) \geq m^*(G_1)\, m^*(G_2)$, $\quad f(G) \geq f(G_1) + f(G_2) - 2.$

We say that G has a core if a graph G' with $|V(G')| \geq 6$ and no non-trivial cuts of size at most 3 can be obtained from G by a (possibly empty) sequence of cut contractions.

Main technical statement

Theorem: Let G be a cubic bridgeless graph then, if G has a core

$$m^*(G) \geq 2^{\alpha|V(G)| - \beta f(G) + \gamma},$$

where $\alpha << \beta << \gamma << 1.$

Main technical statement

Theorem: Let G be a cubic bridgeless graph then, if G has a core

$$m^*(G) \geq 2^{\alpha|V(G)|-\beta f(G)+\gamma}.$$

Sketch of a proof: By induction.

1. If G has no non-trivial cuts of size at most 3 apply Voorhoeve's splitting trick.

$$3m^*(G) \geq m^*(G_1)+m^*(G_2)+m^*(G_3)+m^*(G_4)$$

$$|V(G_i)|=|V(G)|-2$$

$$f(G_i) \leq f(G)+2$$

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Theorem: Let G be a cubic bridgeless graph then, if G has a core

$$m^*(G) \geq 2^{\alpha|V(G)| - \beta f(G) + \gamma}.$$

Sketch of a proof: By induction.

1. If G has no non-trivial cuts of size at most 3 apply Voorhoeve's splitting trick.

2. Easy if for some small cut both contractions G_1 and G_2 have a core

$$m^*(G) \geq m^*(G_1)\, m^*(G_2) \geq 2^{\alpha|V(G)| - \beta f(G) - 2\beta + 2\gamma}$$

$$f(G) \geq f(G_1) + f(G_2) - 2.$$

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Sketch of a proof: By induction.

1. If G has no non-trivial cuts of size at most 3 apply Voorhoeve's splitting trick.

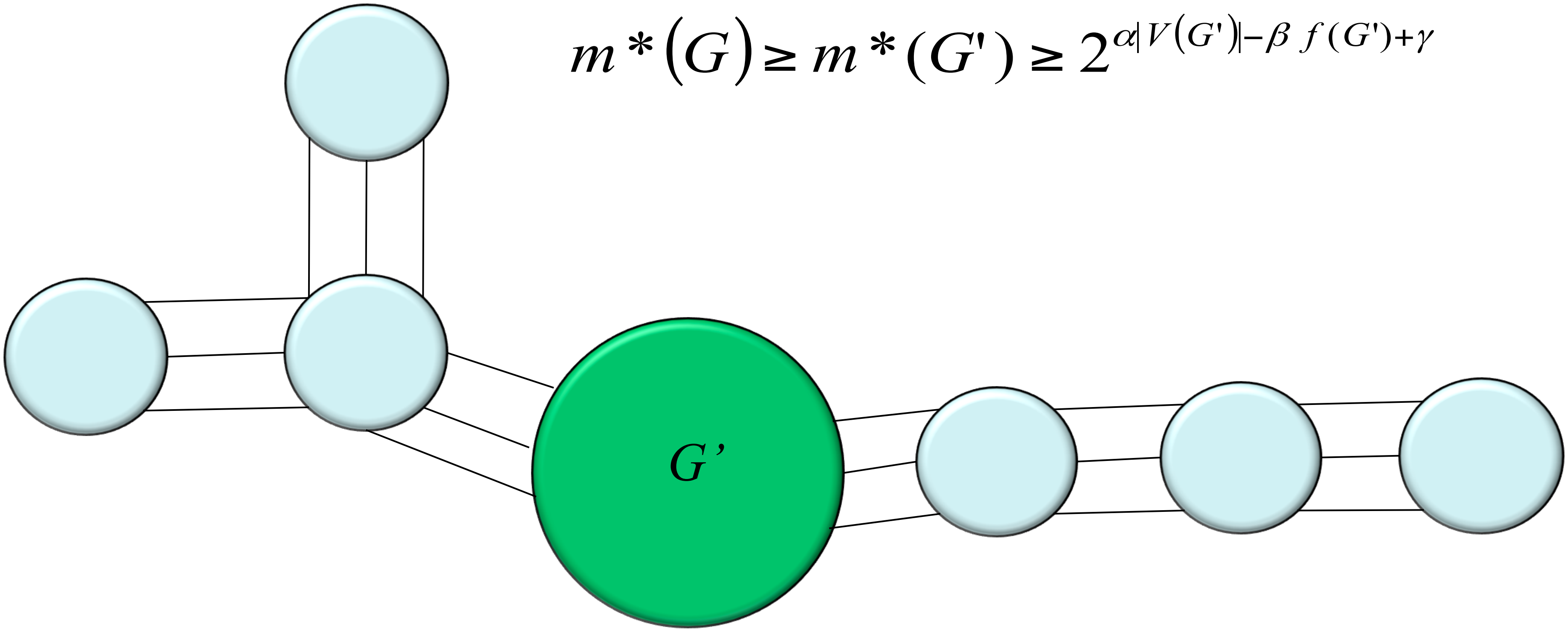
2. Easy if for some small cut both contractions G_1 and G_2 have a core.

3. Otherwise, G has a tree structure with respect to small cuts with exactly one "large" piece.

Cut decomposition

G has a tree structure with respect to small cuts with exactly one “large” piece.

$$m^*(G) \geq m^*(G') \geq 2^{\alpha|V(G')|-\beta f(G')+\gamma}$$



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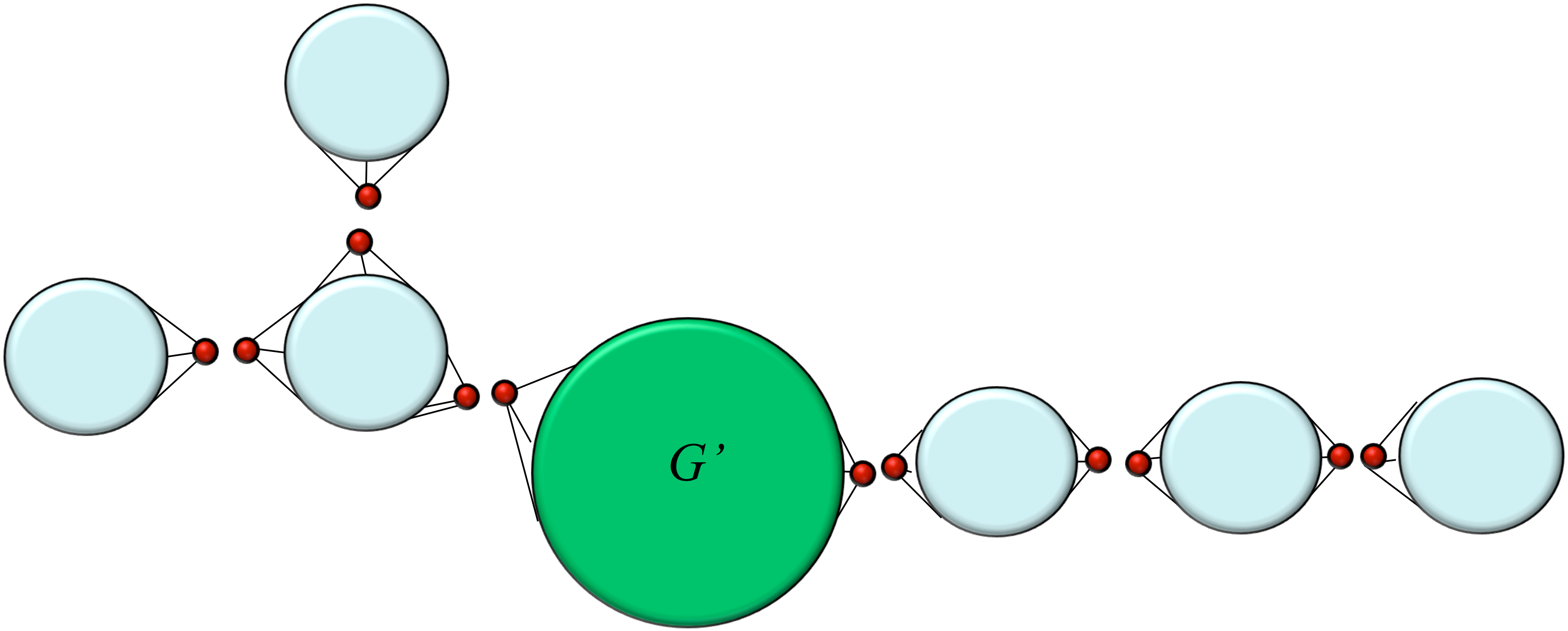

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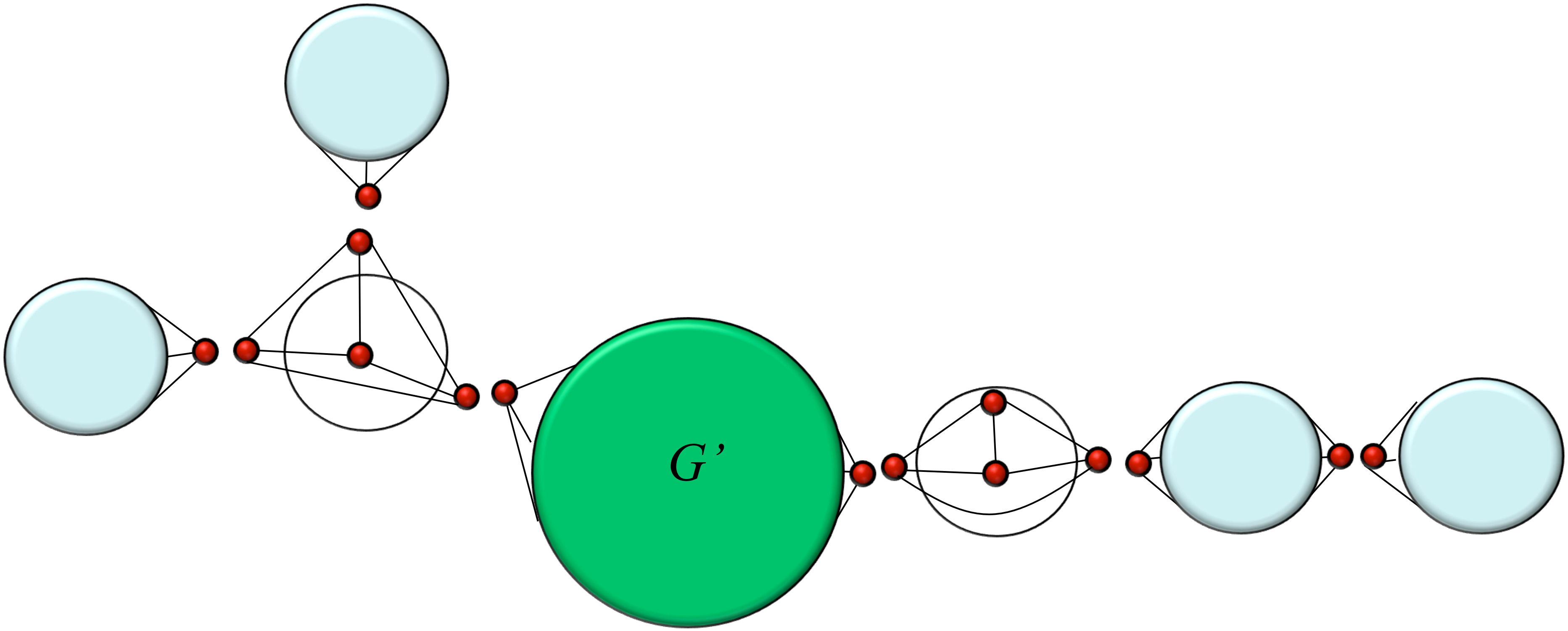

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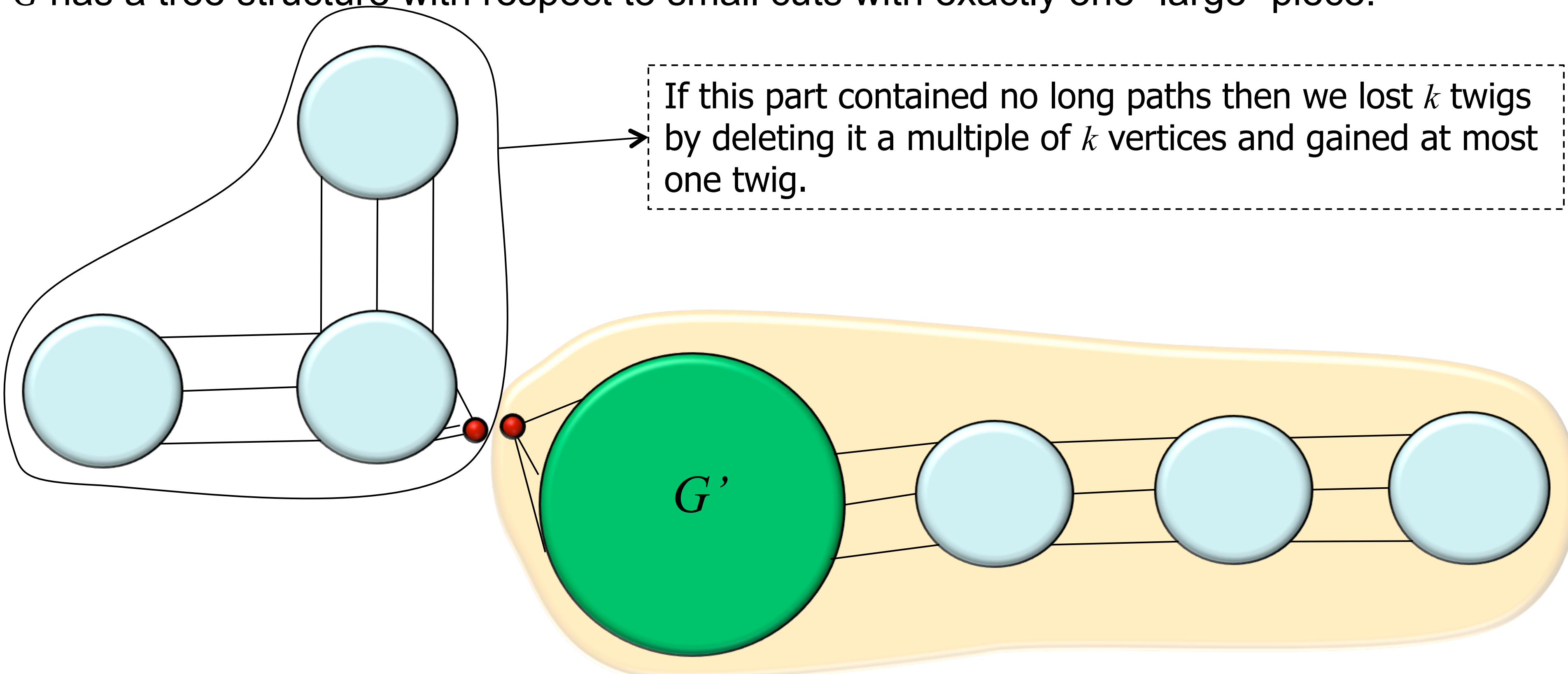
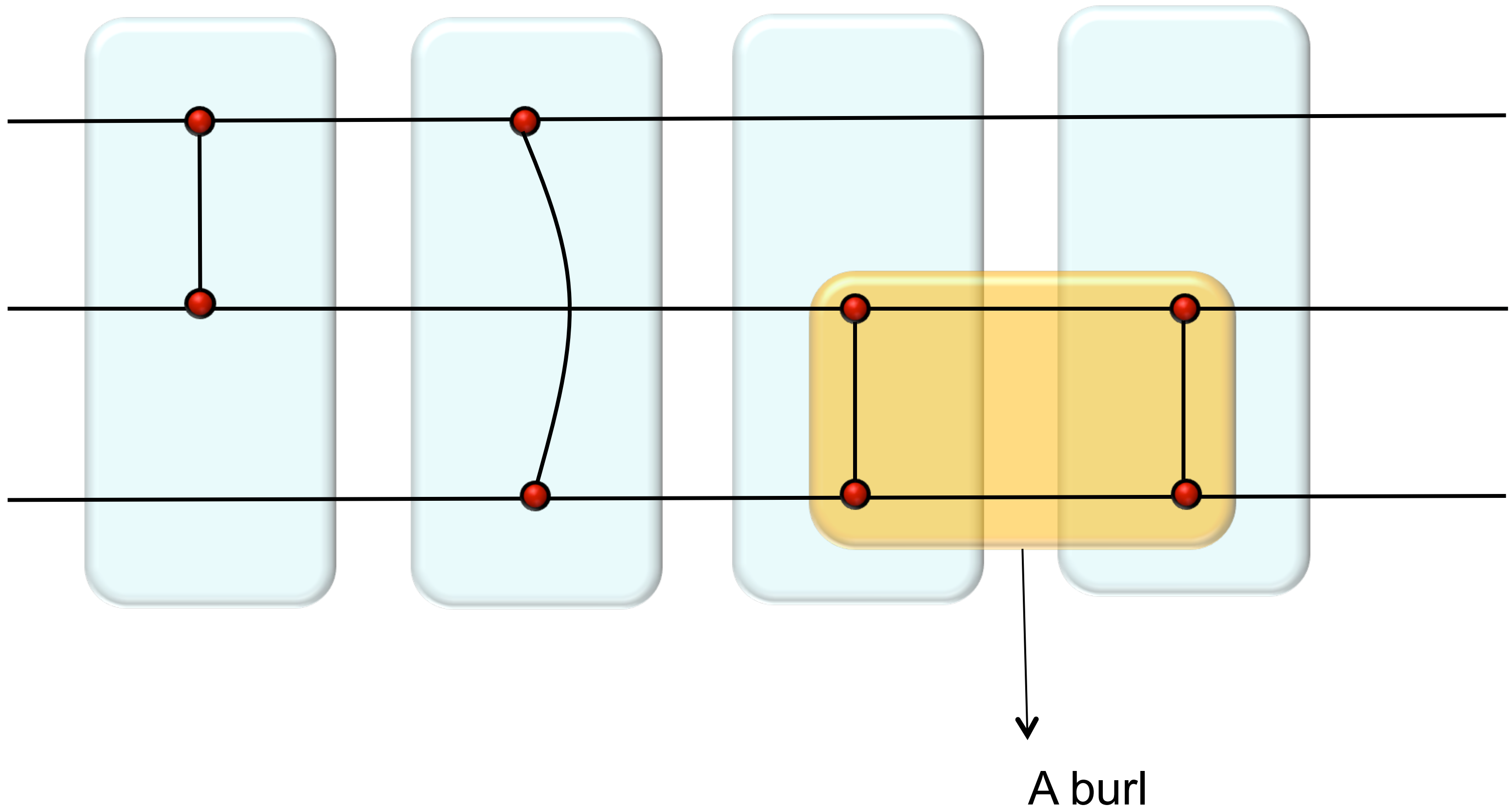


Burls in long paths of 3-cuts



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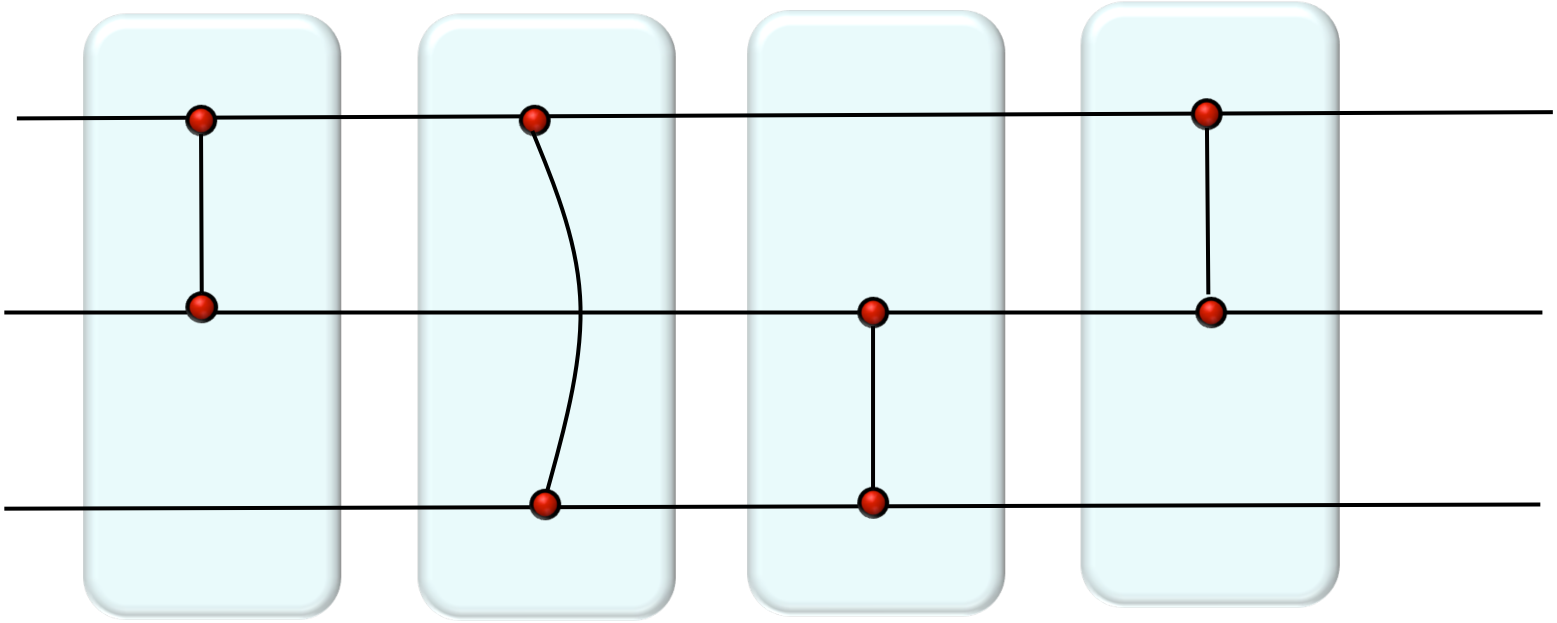

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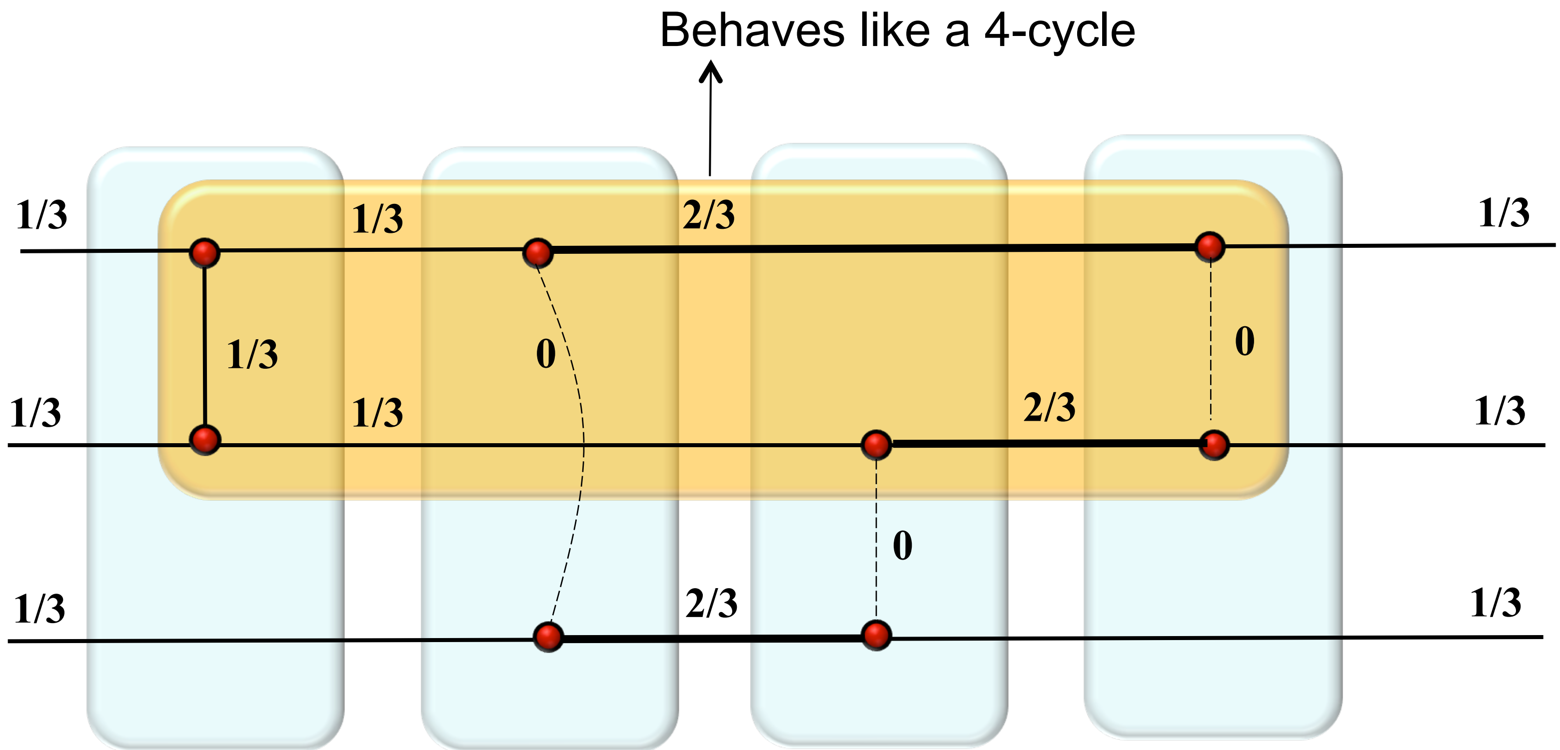

k-regular graphs

Conjecture(Lovász,Plummer,1986): There exist constants $c_1(k)$, $c_2(k)>0$ such that for every k-regular graph G with $m^*(G) \geq 1$, we have

$$m(G) \geq c_1(k)\,(c_2(k))^{|V(G)|}$$

Moreover, $c_2(k) \to 1$ as $k \to 1$.

Counterexample (Geelen, N.): For $k \geq 4$ there exist k-regular graphs G with $m^*(G) \geq 1$, and

$$m(G) \leq 2^{O(\sqrt{|V(G)|})}$$

(Examples are not $(k\text{-}1)$-edge-connected.)

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Conjecture(Lovász,Plummer,1986): There exist constants $c_1(k)$, $c_2(k)>0$ such that for every k-regular $(k\text{-}1)$-edge-connected graph G we have

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Moreover, $c_2(k) \to 1$ as $k \to 1$.

Theorem(Seymour): There exist a constant $\varepsilon > 0$ such that $m(G) \geq 2^{\varepsilon |V(G)|}$ in every k-regular $(k\text{-}1)$-edge-connected graph G.

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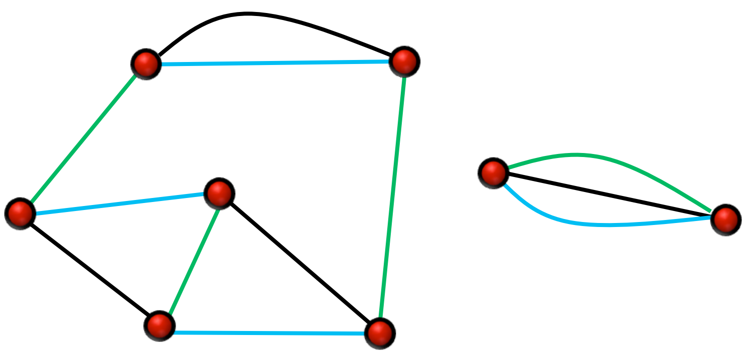
Proof: Consider $w \equiv 1/k \in PMP(G)$.
Choose 3-perfect matchings independently from the corresponding distribution.



Thank you!