MATH 550: Combinatorics. Winter 2018.

Assignment # 3: Discrete Geometry.

Due by e-mail (sent to snorine@gmail.com) by Tuesday, April 17th.

1. For $X \subseteq \mathbb{R}^d$ define $S(X)$ as a set of all points which lie on segments with ends in X. Let $S_2(X) := S(S(X))$ and, more generally, $S_{k+1}(X) = S(S_k(X))$. Show that $S_{\lceil \log_2(d+1) \rceil}(X)$ is always convex.

2. Colored Radon Theorem. Let $A_1, A_2, \ldots, A_{d+1} \in \mathbb{R}^d$ be such that $|A_i| = 2$ for every $i \in [d+1]$. Show that there exist disjoint $X, Y \subseteq A_1 \cup A_2 \cup \ldots \cup A_{d+1}$ such that $|X \cap A_i| = |Y \cap A_i| = 1$ for every $i \in [d+1]$, and $\mathrm{conv}(X) \cap \mathrm{conv}(Y) \neq \emptyset$.

3.

(a) Show that if $x, y, z \in \mathbb{R}^2$ are three points at pairwise distance at most 1 then there exists a disk in $\mathbb{R}^2$ of radius $1/\sqrt{3}$ containing x, y and z.

(b) Show that if $X \subseteq \mathbb{R}^2$ is a finite set of diameter at most 1 then X is contained in some disk of radius $1/\sqrt{3}$.

(c) Find the minimum c such that every finite set of diameter at most 1 in $\mathbb{R}^3$ is contained in some ball of radius c.

4. Matoušek. 1.3.4. A *strip of width w* is a part of the plane bounded by two parallel lines at distance w. The *width* of a set $X \subseteq \mathbb{R}^2$ is the smallest width of a strip containing X.

(a) Show that every compact convex set of width 1 contains a segment of length 1 in every direction.

(b) Let $C_1, C_2, \ldots, C_n$ be closed convex sets in the plane, $n \geq 3$, such that the intersection of every 3 of them has width at least 1. Show that $\cap_{i=1}^n C_i$ has width at least 1.

5. Matoušek. 4.1.5 (a). Use the Szemerédi-Trotter theorem to show that n points in the plane determine at most $O(n^{7/3})$ triangles of unit area.

6. Tao-Vu. 8.2.6. (Beck's theorem.) Let $P \subseteq \mathbb{R}^2$ be finite. Show that there either exists a line incident with $\Omega(|P|)$ points in P or there exist $\Omega(|P|^2)$ lines incident with at least 2 points in P.