MATH 550: Combinatorics. Winter 2018.

Assignment # 2: Turán- and Ramsey-type problems.

Due in class on Thursday, March 29th.

1. Let G be a graph on n vertices for some $n \geq 3$ with $|G| \geq \lfloor \frac{n^2}{4} \rfloor + 1$.

a) Show that G contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.

b) Show that the bound in a) is tight: For every $n \geq 3$ there exists a graph G on n vertices with $|G| = \lfloor \frac{n^2}{4} \rfloor + 1$ containing exactly $\lfloor \frac{n}{2} \rfloor$ triangles.

2. Let $K_{s,s,s}$ denote the 3-graph, whose vertices can be partitioned into three sets A_1, A_2 and A_3, such that $|A_i| = s$ for $i = 1, 2, 3$, and the edges are all the triples $\{x_1, x_2, x_3\}$ such that $x_i \in S_i$ for $i = 1, 2, 3$. Show that $\pi(K_{s,s,s}) = 0$ for every s.

3. **Bollobás. 8.7.** Let $K_4^{(3)}$ denote the complete 3-graph on 4 vertices, i.e. the 3-graph isomorphic to $[4]^{(3)}$. Following de Caen (1983), we give an upper bound on $\pi(K_4^{(3)})$. Let $\mathcal{F} \subseteq [n]^{(3)}$ be a hypergraph containing no $K_4^{(3)}$ with $|\mathcal{F}| = m$. For $x, y \in [n]$, $x \neq y$ let

$$A(x, y) := \{z \in [n] \mid \{x, y, z\} \in \mathcal{F}\},$$

and let $a_{xy} := |A(x, y)|$. Note that if $\{x, y, z\} \in \mathcal{F}$ then $A(x, y) \cap A(y, z) \cap A(z, x) = \emptyset$ and so

$$a_{xy} + a_{yz} + a_{zx} \leq 2n - 3.$$

Summing over all edges of $\mathcal{F}$ deduce that

$$\sum_{\{x,y\} \in [n]^{(2)}} a_{xy}^2 \leq (2n - 3)m.$$

Using convexity of x^2 show that the left hand side is at least $(3m)^2 / \binom{n}{2}$ and deduce that $m \leq \frac{2n-3}{9} \binom{n}{2}$ and $\pi(K_4^{(3)}) \leq 2/3$.

4. Let G be a graph with $V(G) = [17]$ and $x, y \in V(G)$ adjacent if and only if

$$(x - y) \bmod 17 \in \{\pm 1, \pm 2, \pm 4, \pm 8\}.$$

a) Show that neither G nor the complement of G contains a K_4 subgraph.

b) Deduce that $R(4, 4) = 18$.

5. **Schur's theorem.** Show that for every positive integer k there exists a positive integer n satisfying the following. In every coloring of $[n]$ with k colors it is possible to find a triple of (not necessarily distinct) integers x, y, z of the same color so that $x + y = z$.
(*Hint*: Use Ramsey's theorem.)

6. Show that for each $\varepsilon > 0$ there exists N with the following property. For each real $\alpha > 0$ there exist integers q and p such that $1 \leq q \leq N$ and

$$|q^2 \alpha - p| \leq \varepsilon.$$

(*Hint:* Use van der Waerden's theorem.)