MATH 550: Combinatorics. Winter 2015.

Assignment # 2: Turán- and Ramsey-type problems.

Due in class on Wednesday, March 16th.

1. Let G be a graph on n vertices for some $n \geq 3$ with $|G| \geq \lfloor \frac{n^2}{4} \rfloor + 1$.

a) Show that G contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.

b) Show that the bound in a) is tight: For every $n \geq 3$ there exists a graph G on n vertices with $|G| = \lfloor \frac{n^2}{4} \rfloor + 1$ containing exactly $\lfloor \frac{n}{2} \rfloor$ triangles.

2. Let H denote the 3-uniform hypergraph on 6 vertices and 8 edges with

$$V(H) = \{a_1, a_2, b_1, b_2, c_1, c_2\}$$

and

$$E(H) = \{\{a_i, b_j, c_k\} \mid 1 \leq i, j, k \leq 2\}.$$

Show that $\pi(H) = 0$. (For every $\varepsilon > 0$ there exists n_0 such that if G is a 3-uniform hypergraph on $n \geq n_0$ vertices, containing no copy of H, then G has at most $\varepsilon\binom{n}{3}$ edges.)

3. **Bollobás. 8.7.** Let $K_4^{(3)}$ denote the complete 3-graph on 4 vertices, i.e. the 3-graph isomorphic to $[4]^{(3)}$. Following de Caen (1983), we give an upper bound on $\pi(K_4^{(3)})$. Let $\mathcal{F} \subseteq [n]^{(3)}$ be a hypergraph containing no $K_4^{(3)}$ with $|\mathcal{F}| = m$. For $x, y \in [n]$, $x \neq y$ let

$$A(x,y) := \{z \in [n] \mid \{x,y,z\} \in \mathcal{F}\},$$

and let $a_{xy} := |A(x,y)|$. Note that if $\{x,y,z\} \in \mathcal{F}$ then $A(x,y) \cap A(y,z) \cap A(z,x) = \emptyset$ and so

$$a_{xy} + a_{yz} + a_{zx} \leq 2n - 3.$$

Summing over all edges of $\mathcal{F}$ deduce that

$$\sum_{\{x,y\} \in [n]^{(2)}} a_{xy}^2 \leq (2n-3)m.$$

Using convexity of x^2 show that the left hand side is at least $(3m)^2/\binom{n}{2}$ and deduce that $m \leq \frac{2n-3}{9}\binom{n}{2}$ and $\pi(K_4^{(3)}) \leq 2/3$.

4. Let G be a graph with $V(G) = [17]$ and $x, y \in V(G)$ adjacent if and only if

$$(x - y) \bmod 17 \in \{\pm 1, \pm 2, \pm 4, \pm 8\}.$$

a) Show that neither G nor the complement of G contains a K_4 subgraph.

b) Deduce that $R(4,4) = 18$.

5. Hypergraph Ramsey theorem. Show that for all positive integers r, k_1 and k_2 there exists a positive integer $n = R^{(r)}(k_1, k_2)$ so that the following holds. If elements of $[n]^{(r)}$ are colored in colors red and blue then there is a set $Z \subseteq [n]$ such that either $|Z| = k_1$ and all elements of $Z^{(r)}$ are red, or $|Z| = k_2$ and all elements of $Z^{(r)}$ are blue.
(*Hint*: Use induction on r, and, for given r, induction on $k_1 + k_2$. Consider all edges containing a given vertex and attempt to imitate the proof of Ramsey's theorem.)

6. Show that for each $\varepsilon > 0$ there exists N with the following property. For each real $\alpha > 0$ there exist integers q and p such that $1 \leq q \leq N$ and

$$|q^2\alpha - p| \leq \varepsilon.$$

(*Hint:* Use van der Waerden's theorem.)