MATH 550: Combinatorics. Winter 2015.

Assignment #1: Set systems. Due in class on Tuesday, February 17th.

1. **Bollobás 3.9.** Suppose $\mathcal{A} \subseteq \mathcal{P}([n])$ is an *ideal*, i.e. if $B \subseteq A$ and $A \in \mathcal{A}$ then $B \in \mathcal{A}$. Use the local LYM inequality to show that the average size of an element of $\mathcal{A}$ is at most $n/2$.

2. Let n be a positive integer. Consider a set $\mathcal{T}_n = \{0, 1, 2\}^n$ consisting of all sequences $(a_1, a_2, \ldots, a_n)$ with $a_i \in \{0, 1, 2\}$ for $i \in [n]$.

We define a partial order on $\mathcal{T}_n$ so that $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ if and only if $a_i \leq b_i$ for every $i \in [n]$. (For example $(1, 0, 1) \leq (1, 2, 2)$, while $(1, 0, 1)$ and $(0, 1, 2)$ are incomparable.)

For a sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ define the *weight* of $\mathbf{a}$ to be $w(\mathbf{a}) := a_1 + a_2 + \ldots + a_n$. A chain $\mathcal{C} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k)$ with $\mathbf{a}_1 < \mathbf{a}_2 < \ldots < \mathbf{a}_k$ in $\mathcal{T}_n$ is called *symmetric* if $w(\mathbf{a}_{i+1}) = w(\mathbf{a}_i) + 1$ for $i = 1, 2, \ldots, k-1$ and $w(\mathbf{a}_1) + w(\mathbf{a}_k) = 2n$.

a) Show that $\mathcal{T}_n$ allows a symmetric chain decomposition.

b) Give an example of an antichain in $\mathcal{T}_n$ which intersects every symmetric chain. Deduce that this antichain is maximum. (An *antichain* is a subset $\mathcal{A} \in \mathcal{T}_n$ such that for $\mathbf{a}, \mathbf{b} \in \mathcal{A}$ if $\mathbf{a} \leq \mathbf{b}$ then $\mathbf{a} = \mathbf{b}$, i. e. no two distinct elements of $\mathcal{A}$ are comparable.)

3. Let p be a prime and $n < p$ a positive integer. Show that for any $x_1, x_2, \ldots, x_n \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$ and any $x \in \mathbb{Z}/p\mathbb{Z}$, the number of subsets $A \in \mathcal{P}([n])$ such that $\sum_{i \in A} x_i = x$ is at most $\binom{n}{\lfloor n/2 \rfloor}$. (*Hint*: Define sparse set system appropriately and emulate Kleitman's solution to the Littlewood-Offord problem.)

4. **Hilton, 1974.** Let $1 \leq g \leq h \leq n$ be integers with $g + h \leq n$. Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be an intersecting family and suppose that $g \leq |F| \leq h$ for every $F \in \mathcal{F}$. Use Erdős-Ko-Rado theorem to show that

$$|\mathcal{F}| \leq \sum_{i=g}^{h} \binom{n-1}{i-1}.$$

5. A *k-sunflower* in a set system $\mathcal{F}$ on X is a collection of distinct sets $F_1, F_2, \ldots, F_k \in \mathcal{F}$ such that for some $Z \subseteq X$ we have $F_i \cap F_j = Z$ for all $1 \leq i < j \leq k$. (I.e. the intersection of every pair of distinct sets in the sunflower is the same.) Let $c(k, r)$ denote the maximum possible size of a set system $\mathcal{F}$ such that

$(*)$ $|F| \leq r$ for every $F \in \mathcal{F}$, and $\mathcal{F}$ does not contain a k-sunflower.

Suppose that a set system $\mathcal{F}$ on X satisfies $(*)$.

a) Show that there exists a set $Y \subseteq X$ with $|Y| \leq (k-1)r$ such that every set in $\mathcal{F}$ contains an element of Y.

b) Let $\mathcal{F}_y = \{F - y \mid F \in \mathcal{F}, y \in F\}$. Show that $|\mathcal{F}_y| \leq c(k, r-1)$ for every y.

c) Deduce from a) and b) that

$$c(k, r) \leq (k-1)^r r!$$

d) Construct an explicit example of a family $\mathcal{F}$ satisfying $(*)$ to show that

$$c(k, r) \geq (k-1)^r.$$

6. Let $r \geq 1$ be an integer, $\mathcal{A} \subseteq X^{(r)}$ and $i, j \in X$. Write down a detailed proof of the inequality

$$|\partial \tilde{R}_{ij}(\mathcal{A})| \leq |\partial \mathcal{A}|.$$

7. What is the minimum size of compressed $\mathcal{A} \subseteq \mathbb{N}^{(3)}$ such that $\{1, 10, 100\}, \{1, 20, 50\} \in \mathcal{A}$?