MATH 550: Combinatorics. Winter 2015.

Final exam.
Due electronically at snorin@math.mcgill.ca
by 5PM on Thursday, April 30th.

1. Turan-type problems.

Let G be a graph on $n \geq 5$ vertices and let $w : E(G) \to \mathbb{Z}_+$ be a weight function on edges of G. Suppose that for every set $S \subseteq V(G)$ with $|S| = 5$ the sum of the weights of the edges of G with both ends in S is at most 32.

a) Show that the sum of the weights of all edges of G is at most $3\binom{n}{2} + n - 2$.

b) Show that for every $n \geq 5$ there exist G, w satisfying the conditions above such that the sum of the weights of all edges of G is at least $3\binom{n}{2} + (n-1)/2$.

2. Ramsey Theorem.

For a pair of positive integers $n \geq t$ let $d(n,t)$ denote the minimum number of monochromatic copies of K_t among all edge colorings of K_n in two colors.

a) Show that $d(t) := \lim_{n\to\infty} \frac{d(n,t)}{\binom{n}{t}}$ exists for every t and is positive.

b) Show that $d(t) \leq 2^{1-\binom{t}{2}}$ for every $t \geq 2$.

c) Show that $d(3) = 1/4$.

d) Show that $d(4) \geq 2^{-10}$.

3. Convexity.

a) Show that if $x, y, z \in \mathbb{R}^2$ are three points at pairwise distance at most 1 then there exists a disk in $\mathbb{R}^2$ of radius $1/\sqrt{3}$ containing x, y and z.

b) Show that if $X \subseteq \mathbb{R}^2$ is a finite set of diameter at most 1 then X is contained in some disk of radius $1/\sqrt{3}$.

c) Find the minimum c such that every finite set of diameter at most 1 in $\mathbb{R}^3$ is contained in some ball of radius c.

4. Szemeredi-Trotter theorem.

Show that there exists a constant $C > 0$ so that for any finite set $P \subseteq \mathbb{R}^2$ one has

$$|\{(a,b) \mid a, b \in P,\ \|a - b\| = 1\}| \leq C|P|^{4/3}.$$

(*Hint:* Modify Szemeredi-Trotter theorem so that it applies to circles instead of lines.)

5. Combinatorial Nullstellensatz.

Let G be a graph containing a Hamiltonian cycle. Suppose that every vertex $v \in V(G)$ is assigned a set $S(v)$ of two distinct real numbers. Show that it is possible to choose a number $c(v) \in S(v)$ for every vertex $v \in V(G)$, so that $\sum_{w \in N(v)} c(w) \neq 0$ for every $v \in V(G)$.

(A *Hamiltonian cycle* is a cycle containing every vertex of the graph. We denote by $N(v)$ the set of all vertices adjacent to the vertex v.)

6. Shannon capacity.

Let $\Theta(C_9)$ denote the Shannon capacity of the cycle of length 9. Show that

$$\sqrt{18} \leq \Theta(C_9) \leq \frac{9}{2}$$