Problem Solving Seminar Fall 2022. Problem Set 6: Probability.

Classical results.

1. **Monty Hall problem.** Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1 (but the door is not opened), and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

2. What is the probability that three randomly chosen points on a circle form an acute triangle?

3. If a needle of length 1 is dropped at random on a surface ruled with parallel lines at distance 2 apart, what is the probability that the needle will cross one of the lines?

Problems.

1. **Putnam 1968. B1.** The temperatures in Chicago and Detroit are x° and y°, respectively. These temperatures are not assumed to be independent; namely, we are given:

   (i) $P(x^\circ = 70^\circ)$, the probability that the temperature in Chicago is 70°,
   (ii) $P(y^\circ = 70^\circ)$, and
   (iii) $P(\max(x^\circ, y^\circ) = 70^\circ)$.

   Determine $P(\min(x^\circ, y^\circ) = 70^\circ)$.

2. **Putnam 1961. B2.** Two points are selected independently and at random from a segment length α. What is the probability that they are at least distance $\beta(< \alpha)$ apart?

3. **AIMC 1995.** Find the probability that in the process of repeatedly flipping a fair coin, one will encounter a run of 5 heads before one encounters a run of 2 tails.

4. **Putnam 2014. A4.** Suppose X is a random variable that takes on only nonnegative integer values, with $E[X] = 1$, $E[X^2] = 2$, and $E[X^3] = 5$. (Here $E[y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X = 0$.

5. **Putnam 2002. B4.** An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an odd number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you *must* guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

6. **Putnam 2021. B6.** Given an ordered list of $3N$ real numbers, we can *trim* it to form a list of N numbers as follows: We divide the list into N groups of 3 consecutive numbers, and within each group, discard the highest and lowest numbers, keeping only the median.

   Consider generating a random number X by the following procedure: Start with a list of 3^{2021} numbers, drawn independently and uniformly at random between 0 and 1. Then trim this list as defined above, leaving a list of 3^{2020} numbers. Then trim again repeatedly until just one number remains; let X be this number. Let μ be the expected value of $|X - \frac{1}{2}|$. Show that

$$\mu \geq \frac{1}{4}\left(\frac{2}{3}\right)^{2021}.$$