Problem Solving Seminar Fall 2022. Problem Set 3: Inequalities.

Classical results.

1. **AM-GM.** For any non-negative real numbers $x_1, x_2, \ldots, x_n$,

$$\sqrt[n]{x_1 x_2 \ldots x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

2. **Jensen.** For any convex function f and any real $x_1, x_2, \ldots, x_n$,

$$f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}.$$

3. **Cauchy-Schwarz.** For any real $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$,

$$(x_1 y_1 + x_2 y_2 + \ldots + x_n y_n)^2 \leq (x_1^2 + x_2^2 + \ldots x_n^2)(y_1^2 + y_2^2 + \ldots + y_n^2).$$

4. **Arithmetic-harmonic mean.** For any non-negative real numbers $x_1, x_2, \ldots, x_n$,

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

Problems.

1. Show that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$

2. **Putnam 2019. A1.** Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B, and C are nonnegative integers.

3. **Putnam 2003. A2.** Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.$$

4. **Putnam 2021. B2.** Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences $a_1, a_2, a_3, \cdots$ of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1.$$

5. **Putnam 2004. B2.** Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

6. **IMO 1994.** Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \leq n$ for some i, j (possibly the same) we have $a_i + a_j = a_k$ for some k. Prove that:

$$\frac{a_1 + a_2 + \ldots + a_m}{m} \geq \frac{n+1}{2}.$$

7. **Putnam 2003. A4.** Let a, b, c, A, B, C be real, a, A non-zero such that $|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$ for all real x. Show that $|b^2 - 4ac| \leq |B^2 - 4AC|$.