Problem Solving Seminar Fall 2022. Problem Set 8: Geometry.

Classical results.

1. **Triangle area.** Let ABC be a triangle with side lengths $a = BC$, $b = CA$, and $c = AB$, and let r be its inradius and R be its circumradius. Let $s = (a + b + c)/2$ be its semiperimeter. Then its area is
$$sr = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = \frac{1}{2}ab\sin C.$$
2. Every polygon (not necessarily convex) has a triangulation.
3. **Art Gallery.** The floor plan of a single- floor art gallery can be considered as a (not necessarily convex) polygon with n vertices. Prove that it is always possible to position $\lfloor \frac{n}{3} \rfloor$ such that every point inside the gallery has a line-of-sight connection to some guard.
4. **Pick.** The area of any polygon with integer vertex coordinates is exactly $I + B/2 - 1$, where I is the number of lattice points in its interior, and B is the number of lattice points on its boundary.

Problems.

1. **Putnam 1998. A1.** A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?
2. **Putnam 2008. B1.** What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)
3. **Putnam 1955. A2.** O is the center of a regular n-gon $P_1P_2 \dots P_n$ and X is a point outside the n-gon on the line OP_1. Show that $|XP_1| \cdot |XP_2| \cdot \dots \cdot |XP_n| + |OP_1|^n = |OX|^n$.
4. **Putnam 2012. B2.** Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property: If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.
5. **Putnam 2016. B3.** Suppose that S is a finite set of points in the plane such that the area of triangle $\triangle ABC$ is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.
6. **Putnam 1957. A5.** Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are at distance 1 apart.
7. **Putnam 2000. A5.** Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.
8. **Putnam 1958. A7.** Show that we cannot place 10 unit squares in the plane so that no two have an interior point in common and one has a point in common with each of the others.