Problem Solving Seminar Fall 2021. Problem Set 7: Geometry

Classical results.

1. **Triangle area.** Let ABC be a triangle with side lengths $a = BC$, $b = CA$, and $c = AB$, and let r be its inradius and R be its circumradius. Let $s = (a + b + c)/2$ be its semiperimeter. Then its area is
$$sr = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = \frac{1}{2}ab\sin C.$$

2. Every polygon (not necessarily convex) has a triangulation.

3. **Art Gallery.** The floor plan of a single- floor art gallery can be considered as a (not necessarily convex) polygon with n vertices. Prove that it is always possible to position $\lfloor \frac{n}{3} \rfloor$ such that every point inside the gallery has a line-of-sight connection to some guard.

4. **Pick.** The area of any polygon with integer vertex coordinates is exactly $I + B/2 - 1$, where I is the number of lattice points in its interior, and B is the number of lattice points on its boundary.

Problems.

1. **Putnam 1998. A1.** A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

2. **Putnam 1967. B1.** Let hexagon $ABCDEF$ be inscribed in to a circle of radius 1. Show that if $|AB| = |CD| = |EF| = 1$ then the midpoints of BC, DE and FA are the vertices of an equilateral triangle.

3. **Putnam 2008. B1.** What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

4. **Putnam 1955. A2.** O is the center of a regular n-gon $P_1P_2 \ldots P_n$ and X is a point outside the n-gon on the line OP_1. Show that $|XP_1| \cdot |XP_2| \cdot \ldots \cdot |XP_n| + |OP_1|^n = |OX|^n$.

5. **Putnam 2012. B2.** Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property: If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.

6. **USA 2000.** A sequence of polygons is derived as follows. The first polygon is a regular hexagon of area 1. Thereafter each polygon is derived from its predecessor by joining two adjacent edge midpoints and cutting off the corner. Show that all the polygons have area greater than $1/3$.

7. **Putnam 2016. B3.** Suppose that S is a finite set of points in the plane such that the area of triangle $\triangle ABC$ is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.

8. **Putnam 2000. A5.** Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.