Problem Solving Seminar Fall 2021. Problem Set 7: Calculus.

Classical results.

1. Every continuous mapping of a circle into a line carries some pair of diametrically opposite points to the same point.

2. **Leibniz formula.**
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$$

3. **Gaussian integral.**
$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Problems.

1. **Putnam 1964. B1.** Let $a_1, a_2, \dots$ be positive integers such that $\sum_{i=1}^{\infty} \frac{1}{a_i}$ converges. For each n, let b_n denote the number of positive integers i for which $a_i \le n$. Prove that $\lim_{n\to\infty} \frac{b_n}{n} = 0$.

2. **Putnam 2012. B1.** Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

   (i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S;

   (ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

   (iii) If $f(x)$ and $g(x)$ are in S and $f(x) \ge g(x)$ for all $x \ge 0$, then the function $f(x) - g(x)$ is in S.

   Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

3. **Putnam 1991. B2.** Suppose f and g are non-constant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y,
$$f(x + y) = f(x)f(y) - g(x)g(y),$$
$$g(x + y) = f(x)g(y) + g(x)f(y).$$
   If $f'(0) = 0$ prove that $(f(x))^2 + (g(x))^2 = 1$ for all real x.

4. **Putnam 2007. B2.** Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\, dx = 0$. Prove that for every $\alpha \in (0, 1)$,
$$\left| \int_0^{\alpha} f(x)\, dx \right| \le \frac{1}{8} \max_{0\le x\le 1} |f'(x)|.$$

5. **Putnam 2013. A3.** Suppose that the real numbers $a_0, a_1, \dots, a_n$ and x, with $0 < x < 1$, satisfy
$$\frac{a_0}{1 - x} + \frac{a_1}{1 - x^2} + \dots + \frac{a_n}{1 - x^{n+1}} = 0.$$
   Prove that there exists a real number y with $0 < y < 1$ such that
$$a_0 + a_1 y + \dots + a_n y^n = 0.$$

6. **Putnam 2008. A4.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \le e \\ xf(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

7. **Putnam 1993. B4.** The function $K(x, y)$ is positive and continuous for $0 \le x \le 1, 0 \le y \le 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \le x \le 1$. Suppose that

$$\int_0^1 f(y)K(x, y)\, dy = g(x) \qquad \text{and} \qquad \int_0^1 g(y)K(x, y)\, dy = f(x)$$

for all $0 \le x \le 1$. Show that $f(x) = g(x)$ for $0 \le x \le 1$.