Problem Seminar.

Number theory.

Classical results.

1. **Euler.** For a positive integer n and any integer a relatively prime to n one has

$$a^{\phi(n)} \equiv 1 \pmod{n},$$

where $\phi(n)$ is the number of positive integers between 1 and n relatively prime to n.

2. **Polignac's formula.** If p is a prime number and n a positive integer, then the exponent of p in $n!$ is

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots.$$

3. **Chinese Remainder theorem.** Let $m_1, m_2, \dots, m_k$ be pairwise positive integers greater than 1, such that $\gcd(m_i, m_j) = 1$ for $i \neq j$. Then for any integers $a_1, a_2, \dots, a_k$ the system of congruences

$$\begin{aligned} x &\equiv a_1 \pmod{m_1}, \\ x &\equiv a_2 \pmod{m_2}, \\ &\dots \\ x &\equiv a_k \pmod{m_k}. \end{aligned}$$

has solutions, and any two such solutions are congruent modulo $m = m_1 m_2 \dots m_k$.

4. **Sylvester's theorem.** Let a and b be positive integers with $\gcd(a, b) = 1$. Then $ab - a - b$ is the largest positive integer c for which the equation $ax + by = c$ is not solvable in nonnegative integers.

Problems.

1. Prove that $n!$ is not divisible by 2^n for any positive integer n.

2. **Putnam 1956. A2.** Given any positive integer n, show that we can find a positive integer m such that mn uses all ten digits when written in the usual base 10.

3. **Putnam 2000. A2.** Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

4. **Putnam 2013. A2.** Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \dots, a_r$ such that $n < a_1 < a_2 < \dots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minumum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

5. **Putnam 2000. B2.** Prove that the expression
$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$
is an integer for all pairs of integers $n \geq m \geq 1$.

6. **USA 1991.** Let n be an arbitrary positive integer. Show that the following sequence is eventually constant modulo n:
$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, 2^{2^{2^{2^2}}}, 2^{2^{2^{2^{2^2}}}}, \ldots$$

7. **IMO 2002.** The positive divisors of an integer $n > 1$ are $1 = d_1 < d_2 < \ldots < d_k = n$. Let $s = d_1d_2 + d_2d_3 + \ldots + d_{k-1}d_k$. Prove that $s < n^2$ and find all n for which s divides n^2.

8. **IMO 2011.** Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.