Problem Seminar. Fall 2017.

Problem Set 10. Miscellaneous.

1. **Putnam 2010. A1.** Given a positive integer n, what is the largest k such that the numbers $1, 2, \dots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.]

2. **Putnam 2010. A2.** Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f'(x) = \frac{f(x+n) - f(x)}{n}$$
for all real numbers x and all positive integers n.

3. **Putnam 2010. B1.** Is there an infinite sequence of real numbers $a_1, a_2, a_3, \dots$ such that
$$a_1^m + a_2^m + a_3^m + \cdots = m$$
for every positive integer m?

4. **Putnam 2010. B2.** Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

5. **Putnam 2010. B3.** There are 2010 boxes labeled $B_1, B_2, \dots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

6. **Putnam 2011. A1.** Define a *growing spiral* in the plane to be a sequence of points with integer coordinates $P_0 = (0, 0), P_1, \dots, P_n$ such that $n \geq 2$ and:
   - the directed line segments $P_0P_1, P_1P_2, \dots, P_{n-1}P_n$ are in the successive coordinate directions east (for P_0P_1), north, west, south, east, etc.;
   - the lengths of these line segments are positive and strictly increasing.

   How many of the points (x, y) with integer coordinates $0 \leq x \leq 2011, 0 \leq y \leq 2011$ *cannot* be the last point, P_n of any growing spiral?

7. **Putnam 2011. A2.** Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \dots$. Assume that the sequence (b_j) is bounded. Prove that
$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \dots a_n}$$
converges, and evaluate S.

8. **Putnam 2011. B1.** Let h and k be positive integers. Prove that for every $\epsilon > 0$, there are positive integers m and n such that

$$\epsilon < |h\sqrt{m} - k\sqrt{n}| < 2\epsilon.$$

9. **Putnam 2011. B2.** Let S be the set of all ordered triples (p, q, r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?

10. **Putnam 2011. B3.** Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?