Problem Seminar. Inequalities.

Classical results.

1. **AM-GM.** For any non-negative real numbers $x_1, x_2, \dots, x_n$,
$$\sqrt[n]{x_1 x_2 \dots x_n} \le \frac{x_1 + x_2 + \dots + x_n}{n}.$$
2. **Cauchy-Schwarz.** For any real $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$,
$$(x_1 y_1 + x_2 y_2 + \dots + x_n y_n)^2 \le (x_1^2 + x_2^2 + \dots x_n^2)(y_1^2 + y_2^2 + \dots + y_n^2).$$
3. **Jensen.** For any convex function f and any real $x_1, x_2, \dots, x_n$,
$$f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \le \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}.$$

Problems.

1. Show that
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$
2. **Putnam 2003. A2.** Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be nonnegative real numbers. Show that
$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \le [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.$$
3. **Putnam 2014. B2.** Suppose that f is a function on the interval $[1, 3]$ such that $-1 \le f(x) \le 1$ for all x and $\int_1^3 f(x)\, dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\, dx$ be?
4. **GA 133.** Let $a, b, c \ge 0, a + b + c = 1$. Prove that
$$0 \le ab + bc + ac - 2abc \le \frac{7}{27}.$$
5. **Putnam 2002. B3.** Show that, for all integers $n > 1$,
$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$
6. **Putnam 2003. B6.** Show that
$$\int_0^1 \int_0^1 |f(x) + f(y)|\, dx\, dy \ge \int_0^1 |f(x)|\, dx$$
for any continuous real-valued function f on $[0, 1]$.