Problem Seminar. Fall 2017.

Problem Set 8. Geometry.

Classical results.

1. **Triangle area.** Let ABC be a triangle with side lengths $a = BC$, $b = CA$, and $c = AB$, and let r be its inradius and R be its circumradius. Let $s = (a + b + c)/2$ be its semiperimeter. Then its area is

$$sr = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = \frac{1}{2}ab\sin C.$$

2. Every polygon (not necessarily convex) has a triangulation.

3. **Art Gallery.** The floor plan of a single- floor art gallery can be considered as a (not necessarily convex) polygon with n vertices. Prove that it is always possible to position $\lfloor \frac{n}{3} \rfloor$ such that every point inside the gallery has a line-of-sight connection to some guard.

4. **Pick.** The area of any polygon with integer vertex coordinates is exactly $I + B/2 - 1$, where I is the number of lattice points in its interior, and B is the number of lattice points on its boundary.

Problems.

1. **Putnam 1999. B1.** Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0} |EF|$.

2. **Putnam 2008. B1.** What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

3. **Putnam 1955. A2.** O is the center of a regular n-gon $P_1P_2\ldots P_n$ and X is a point outside the n-gon on the line OP_1. Show that $|XP_1| \cdot |XP_2| \cdot \ldots \cdot |XP_n| + |OP_1|^n = |OX|^n$.

4. **Putnam 1957. A5.** Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are at distance 1 apart.

5. **Putnam 2012. B2.** Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property: If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.

6. **Putnam 2013. A5.** For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} ($1 \leq i < j < k \leq m$) is said to be *area definite* for $\mathbb{R}^n$ if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \cdot \text{Area}(\Delta A_i A_j A_k) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1$, $a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

7. **Putnam 1991. A4.** Does there exist an infinite sequence of closed discs $D_1, D_2, D_3, \ldots$ in the plane, with centers $c_1, c_2, c_3, \ldots$, respectively, such that

   (a) the c_i have no limit point in the finite plane,

   (b) the sum of the areas of the D_i is finite, and

   (c) every line in the plane intersects at least one of the D_i?

8. **Putnam 2000. A5.** Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

9. **Putnam 1992. A6.** Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points?