Problem Seminar 2017.

Problem Set 5. Algebra.

Classical results.

1. **Hilbert.** Let
$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}.$$
Then $\det(H) \neq 0$.

2. **Lagrange interpolation.** For every positive integer n and every collection of real numbers $a_1, a_2, \ldots, a_n$ there exists a polynomial of degree at most n so that $P(1) = a_1$, $P(2) = a_2, \ldots,$ $P(n) = a_n$.

3. **Cayley-Hamilton.** Given an $n \times n$ matrix A *the characteristic polynomial* of A is defined by $P_A(\lambda) = \det(\lambda I_n - A)$, where I_n is the identity matrix. Then $P_A(A) = 0$ for every A.

4. Let p be a prime. Show that the polynomial $x^{p-1} + x^{p-2} + \ldots + x + 1$ can not be expressed as a product of two non-constant polynomials with integer coefficients.

5. In Oddtown there are n citizens and m clubs $A_1, A_2, \ldots, A_m \subseteq \{1, 2, \ldots, n\}$. The laws of Oddtown prescribe that

   - The clubs must have distinct memberships. ($A_i \neq A_j$ for $i \neq j$),
   - Every club has odd number of members,
   - Every two distinct clubs have an even number of members in common. ($|A_i \cap A_j|$ is even if $i \neq j$).

   Show that $m \leq n$.

Problems.

1. **Putnam 1991. A2.** Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

2. **Putnam 2014. A2.** Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is
$$\frac{1}{\min(i,j)}$$
for $1 \leq i, j \leq n$. Compute $\det(A)$.

3. **Korea. 1999.** Find all functions $f : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R}$ satisfying
$$f\left(\frac{x-3}{x+1}\right) + f\left(\frac{3+x}{1-x}\right) = x$$
for all $x \neq \pm 1$.

4. **Putnam 1976. B2.** Let S be a set with a binary operation $*$ such that

$$a * (a * b) = b \qquad \text{for all } a, b \in S \tag{1}$$
$$(a * b) * b = a \qquad \text{for all } a, b \in S \tag{2}$$

Show that $*$ is commutative, but not necessarily associative.

5. **Putnam 1994. A4.** Let A and B be 2×2 matrices with integer entries such that $A, A + B, A + 2B, A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

6. **Putnam 2007. B4.** Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that
$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$
and $\deg P > \deg Q$.

7. **MIT PS seminar.** A mansion has n rooms. Each room has a lamp and a switch connected to its lamp. However, switches may also be connected to lamps in other rooms, subject to the following condition: if the switch in room a is connected to the lamp in room b, then the switch in room b is also connected to the lamp in room a. Each switch, when flipped, changes the state (from on to off or vice versa) of each lamp connected to it. Suppose at some points the lamps are all off. Prove that no matter how the switches are wired, it is possible to flip some of the switches to turn all of the lamps on. (Hint: interpret as a linear algebra problem over the field of two elements.)

8. **Putnam 1996. B6.** The origin lies inside a convex polygon whose vertices have coordinates (a_i, b_i) for $i = 1, 2, \ldots, n$. Show that we can find $x, y > 0$ such that

$$a_1 x^{a_1} y^{b_1} + a_2 x^{a_2} y^{b_2} + \ldots + a_n x^{a_n} y^{b_n} = 0$$

and
$$b_1 x^{a_1} y^{b_1} + b_2 x^{a_2} y^{b_2} + \ldots + b_n x^{a_n} y^{b_n} = 0.$$