Problem Solving seminar. Team selection contest 2016.

1. In the expansion of $(a+b)^n$, where n is a natural number, there are $n+1$ dissimilar terms. Find the number of dissimilar terms in the expansion of $(a+b+c)^n$.

2.Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that

$$\varepsilon < |h\sqrt{m} - k\sqrt{n}| < 2\varepsilon.$$

3. Show that for every positive rational number q there exists a finite set of (distinct) positive integers S such that $q = \sum_{s\in S} \frac{1}{s}$.

4. A cubical box with sides of length 7 has vertices at $(0,0,0), (7,0,0), (0,7,0), (7,7,0), (0,0,7), (7,0,7), (0,7,7), (7,7,7)$. The inside of the box is lined with mirrors and from the point $(0,1,2)$, a beam of light is directed to the point $(1,3,4)$. The light then reflects repeatedly off the mirrors on the inside of the box. Determine how far the beam of light travels before it first returns to its starting point at $(0,1,2)$.

5. Determine all ordered pairs (m,n) of positive integers such that

$$\frac{n^3+1}{mn-1}$$

is an integer.

6. For $n \geq 2$, a fixed positive integer, find the smallest constant C such that for all nonnegative reals $x_1, \ldots, x_n$,

$$\sum_{1\leq i<j\leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{i=1}^{n} x_i \right)^4 .$$