Problem Seminar. Fall 2016.

Problem Set 8. Miscellaneous.

1. **Putnam 2006. A1.** Find the volume of the region of points (x, y, z) such that
$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

2. **Putnam 2006. A2.** Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

3. **Putnam 2006. B2.** Prove that, for every set $X = \{x_1, x_2, \dots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that
$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

4. **Putnam 2007. A1.** Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

5. **Putnam 2007. A3.** Let k be a positive integer. Suppose that the integers $1, 2, 3, \dots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

6. **Putnam 2007. B1.** Let f be a non constant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

7. **Putnam 2008. A1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

8. **Putnam 2008. A3.** Start with a finite sequence $a_1, a_2, \dots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (*Note: gcd means greatest common divisor and lcm means least common multiple.*)

9. **Putnam 2008. B3.** What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

10. **Putnam 2009. A1.** Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?

11. **Putnam 2009. A2.** Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \quad f(0) = 1, \\ g' &= fg^2h + \frac{4}{fh}, \quad g(0) = 1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

12. **Putnam 2009. B2.** A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?