Problem Seminar. Fall 2016.

Problem Set 3. Induction.

Classical results.

1. Finitely many lines divide the plane into regions. Show that these regions can be colored by two colors in such a way that neighboring regions have different colors.

2. An *Hadamard matrix* is an $n \times n$ square matrix, all of whose entries are -1 or 1, such that every pair of distinct rows is orthogonal. In other words, if the rows are considered to be vectors of length n, then the dot product between any two distinct row-vectors is zero. Show that there exist infinitely many Hadamard matrices.

3. Prove that any positive integer can be represented as $\pm 1^2 \pm 2^2 \pm \ldots \pm n^2$ for some positive integer n and some choice of the signs.

4. Prove that the Fibonacci sequence satisfies the identity
$$F_{2n+1} = F_{n+1}^2 + F_n^2$$
for $n \geq 0$ (The Fibonacci sequence F_n is defined by $F_0 = 1, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.)

Problems.

1. **Put 2005. A1.** Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

2. **Putnam 2001. A2.** You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

3. **Putnam 2003. B2.** Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n. Show that $x_n \leq \frac{2}{n}$.

4. **USA 1997.** An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a silver matrix if, for each $i = 1, 2, \ldots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

   (a) there is no silver matrix for $n = 1997$;

   (b) silver matrices exist for infinitely many values of n.

5. **Putnam 2014. A3.** Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute
$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$
in closed form.

6. **Putnam 2006. B3.** Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

7. **Putnam 2000. B5.** Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a - 1$ or a is in S_n. Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.