Problem Seminar. Fall 2015.
Problem Set 5. Combinatorics.

Classical results.

1. Show that the equation
$$x_1 + x_2 + \ldots + x_r = n$$
has exactly $\binom{n+r-1}{r-1}$ non-negative integer solutions.

2. Consider a convex polygon with n vertices so that no 3 diagonals go through the same point.

   (a) How many intersection points do the diagonals have?

   (b) Into how many regions do the diagonals divide the interior of the polygon?

3. An unfair coin (probability p of showing heads) is tossed n times. What is the probability that the number of heads will be even?

4. **Erdős-Ko-Rado.** Let $\mathcal{F}$ be a family of k element subsets of an n element set, with $n \geq 2k$, such that every two sets in $\mathcal{F}$ have a non-empty intersection. Then
$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Problems.

1. **Putnam 2003. A1.** Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, $2+2$, $1+1+2$, $1+1+1+1$.

2. **Putnam 1996. B1.** Define a *selfish set* to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

3. **Putnam 2001. B1.** Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is
$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n.$$
Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

4. **Germany 1971.** Given 2^{n-1} subsets of a set with n elements with the property that any three have nonempty intersection, prove that the intersection of all the sets is nonempty.

5. **Putnam 2010. B3.** There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

6. **Putnam 1997. A5.** Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

7. **IMO 2014.** Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 squares.

8. **Putnam 1996. B5.** We call a finite string of the symbols X and O *balanced* if every substring of consecutive symbols has a difference of at most 2 between the number of X's and the number of O's. For example, $XOOXOOX$ is not balanced, because the substring $OOXOO$ has a difference of 3. Find the number of balanced strings of length n.