Problem Seminar. Fall 2015.

Problem Set 5. Algebra.

Classical results.

1. **Vandermonde.** Let
$$V = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}.$$
   Then
$$\det(V) = \prod_{1 \le i < j \le n} (x_j - x_i).$$

2. **Lagrange interpolation.** For every positive integer n and every collection of real numbers $a_1, a_2, \dots, a_n$ there exists a polynomial of degree at most n so that $P(1) = a_1$, $P(2) = a_2, \dots, P(n) = a_n$.

3. **Cayley-Hamilton.** Given an $n \times n$ matrix A *the characteristic polynomial* of A is defined by $P_A(\lambda) = \det(\lambda I_n - A)$, where I_n is the identity matrix. Then $P_A(A) = 0$ for every A.

4. **Eisenstein criterion.** Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ be a polynomial with integer coefficients and p be a prime so that

   (i) p divides $a_0, a_1, \dots a_{n-1}$;

   (ii) p does not divide a_n;

   (iii) p^2 does not divide a_0.

   Then $P(x)$ can not be expressed as a product of two non-constant polynomials with integer coefficients.

5. In Oddtown there are n citizens and m clubs $A_1, A_2, \dots, A_m \subseteq \{1, 2, \dots, n\}$. The laws of Oddtown prescribe that

   - The clubs must have distinct memberships. ($A_i \neq A_j$ for $i \neq j$),
   - Every club has odd number of members,
   - Every two distinct clubs have an even number of members in common. ($|A_i \cap A_j|$ is even if $i \neq j$).

   Show that $m \le n$.

Problems.

1. **Putnam 2007. B1.** Let f be a non-constant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

2. **Putnam 1991. A2.** M and N are real unequal $n \times n$ matrices satisfying $M^3 = N^3$ and $M^2 N = N^2 M$. Can we choose M and N so that $M^2 + N^2$ is invertible?

3. **Putnam 2012. A2.** Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

4. **Putnam 2008. A2.** Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

5. **IMO 1993.** Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

6. **Putnam 1994. A4.** Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

7. **Putnam 2006. B4.** Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z$.

8. **Putnam 1996. B6.** The origin lies inside a convex polygon whose vertices have coordinates (a_i, b_i) for $i = 1, 2, \ldots, n$. Show that we can find $x, y > 0$ such that

$$a_1 x^{a_1} y^{b_1} + a_2 x^{a_2} y^{b_2} + \ldots + a_n x^{a_n} y^{b_n} = 0$$

and

$$b_1 x^{a_1} y^{b_1} + b_2 x^{a_2} y^{b_2} + \ldots + b_n x^{a_n} y^{b_n} = 0.$$