Problem Solving seminar. Team selection contest 2014.

1. Recall that the Fibonacci numbers are defined by $F(0) = 0, F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Find the last digit of $F(2014)$ (e.g. the last digit of 2014 is 4.)

2. Let k be the smallest possible integer for which there exist distinct integers $m_1, m_2, \ldots, m_5$ such that the polynomial

$$p(x) = (x - m_1)(x - m_2)\ldots(x - m_5)$$

has exactly k non-zero coefficients. Find, with proof, a set of integers $m_1, m_2, \ldots, m_5$ for which the minimum value k is achieved.

3. Find

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 + 2n}$$

for $|x| < 1$.

4. Let r, s and t be non-negative integers with $t \geq r + s$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \ldots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}$$

5. Show that if the points of the right triangle with side lengths $1, 1$ and $\sqrt{2}$ are colored in four colors, then there must be two points of the same color which are distance at least $2 - \sqrt{2}$ apart.

6. Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.