MATH 350: Graph Theory and Combinatorics. Fall 2015.

Assignment #2: Spanning trees, bipartite graphs and matchings.

1. We say that $F \subseteq E(G)$ is *even-degree* if every vertex of G is incident with an even number of non-loop edges in F. Show that if T is a spanning tree of G, there is an even-degree set $F \subseteq E(G)$ with $F \cup E(T) = E(G)$.

Solution: We claim that if F_1 and F_2 are both even-degree then so is $F_1 \triangle F_2 := (F_1 - F_2) \cup (F_2 - F_1)$. Indeed if E_1 and E_2 are the sets of edges in F_1 and F_2, respectively, incident to the vertex v, then $|E_1 \triangle E_2| = |E_1| + |E_2| - 2|E_1 \cap E_2|$, which is even if $|E_1|$ and $|E_2|$ are even.
For $e \in E(G) - E(T)$, let $F(e)$ be the edge set of the fundamental cycle of e with respect to T. Then $F(e)$ is even-degree. Let

$$F := F(e_1) \triangle F(e_2) \ldots \triangle F(e_k),$$

where $E(G) - E(T) = \{e_1, e_2, \ldots, e_k\}$. Then F is an even-degree set, by the claim above, and $F \cup E(T) = E(G)$, as $e_i \in F(e_i)$ and $e_i \notin F(e_j)$ for $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$.

2. Show that a graph G is bipartite if and only if $\alpha(H) \geq |V(H)|/2$ for every subgraph H of G.

Solution: If G is bipartite and H is a subgraph of G then either $A \cap V(H)$ or $B \cap V(H)$ has size at least $|V(H)|/2$. If G is not bipartite then it contains an odd cycle H and $\alpha(H) = \frac{1}{2}(|V(H)| - 1) < |V(H)|/2$.

3. Let $k \geq 3$ be an integer. Let G be a bipartite graph such that

$$3 \leq \deg(v) \leq k \qquad \text{for every } v \in V(G).$$

Show that G contains a matching of size at least $\frac{3|V(G)|}{2k}$.

Solution: By König's theorem is suffices to show that $\tau(G) \geq \frac{3|V(G)|}{2k}$. Let X be a vertex cover of G. Then the vertices of X are incident to at most $k|X|$ edges in G, and so $|E(G)| \leq k|X|$. On the other hand, $|E(G)| = \frac{1}{2}\sum_{v \in V(G)} \deg v \geq \frac{3}{2}|V(G)|$. Thus $k|X| \geq \frac{3}{2}|V(G)|$, and $|X| \geq \frac{3|V(G)|}{2k}$, as desired.

4. Let G be a bipartite graph with bipartition (A, B) in which every vertex has degree ≥ 1. Assume that for every edge of G with ends $a \in A$ and $b \in B$ we have $\deg(a) \geq \deg(b)$. Show that there exists a matching in G covering A.

Solution: Suppose not. By Hall's theorem there exists $X \subset A$ with $< |X|$ neighbors in B. Choose such a set X with $|X|$ minimum. Let $Y \subset B$ denote the set of vertices adjacent to any of the vertices in X. Then there exists a matching M consisting of edges joining vertices of Y to vertices of X of size $|Y|$. Indeed, otherwise, by Hall's theorem, there exists $Y' \subset Y$ so that the set X' of vertices in X adjacent to any of the vertices of Y' satisfies $|X'| < |Y'|$. It follows that $|X - X'| < |Y - Y'|$ and, as the vertices of $X - X'$ have no neighbors in Y', we deduce that $X - X'$ contradicts the minimality of X.
Let F denote the set of edges joining X and Y. Then

$$|F| = \sum_{a \in A} \deg(a) > \sum_{\substack{e=ab \in M \\ a \in A, b \in B}} \deg(a) \geq$$
$$= \sum_{\substack{e=ab \in M \\ a \in A, b \in B}} \deg(b) \geq |F|,$$

a contradiction. Thus G contains a matching covering A, as desired.

5. Given integers $n \geq m \geq k \geq 0$, determine the maximum possible number of edges in a simple bipartite graph G with bipartition (A, B), with $|A| = n, |B| = m$ and no matching of size k.

Solution: If G has no matching of size k then by König's theorem it contains a set X with $|X| \leq k-1$ so that every edge has an end in X. Every vertex in X is incident with at most n edges. Therefore, $|E(G)| \leq (k-1)n$. One can have a graph with these many edges satisfying all the criteria by having exactly $k - 1$ vertices of B with non-zero degree, each joined to all the vertices of A.