MATH 340: Discrete Structures II. Winter 2017.

Assignment #3: Discrete Probability.

1. *Bayes Theorem.*

a) *Babies.* A family has two children, and at least one of them is a boy. What is the probability that the family has one boy and one girl?

b) *A die and a coin.* Bob rolls a single six-sided die, and then flips a coin the number of times showing on the die. The coin comes up heads every time. What is the probability that the die showed 6?

c) *Drug test.* A large company gives a new employee a drug test. The False-Positive rate is 2% and the False-Negative rate is 10%. In addition, 1% of the population use the drug. The employee tests positive for the drug. What is the probability the employee uses the drug?

Solution:
a): Let A be the that at least one of the children is a boy, and B the event that the family has one boy and one girl. Then $p(A) = 1 - 1/2 * 1/2 = 3/4, p(A \cap B) = 1/2$. Therefore, by definition of the conditional probability $P(B|A) = \boxed{2/3}$.

b): Let A be the event that the die showed 6 and B the event that the coin comes up heads the number of times shown on the die. Then $p(A) = 1/6, p(B/A) = 1/64$

$$p(B) = \frac{1}{6}\left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^6}\right) = \frac{1}{6} \cdot \frac{63}{64}.$$

Therefore

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} = \frac{1/64 \cdot 1/6}{1/6 \cdot 63/64} = \boxed{\frac{1}{63}}.$$

c): Let A be the event that the employee uses the drug, and B the event that employee tests positive for the drug. Then $p(B|A) = 0.9, p(A) = 0.01$ and

$$p(B) = p(A)p(B|A) + p(\bar{A})p(B|\bar{A}) = 0.01 \cdot 0.9 + 0.99 \cdot 0.02 = 0.0288.$$

Thus

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} = \frac{0.009}{0.0288} = \boxed{0.3125}.$$

2. *Monty Hall variant.* Consider an alternative version of the problem where Monty is absent-minded and forgets where the prize is. You select a door. Monty opens one of the other two doors *at random.* If Montys door contains the prize you lose and the game is over. On the other hand, suppose Montys door did not contain the prize. Should you now switch doors when offered the chance?

Solution: Let A_i denote the event that the car is behind Door $\#i$ for $i = 1, 2, 3$, and let M_i denote the event that Monty opens Door $\#i$ for $i = 2, 3$. Then $p(A_i) = 1/3$ for every i, $p(M_j) = 1/2$ for $j = 2, 3$ and $p(M_i \cap A_j) = 1/6$ for all such i and j. Suppose that we selected Door #1 and Monty opens Door #2 and the car is not behind Door #2. We have $p(A_1|(A_1 \cup A_3) \cap M_2) = p(A_1 \cap M_2)/(p(A_1 \cap M_2) + p(A_3 \cap M_2)) = 1/2$, and similarly $p(A_3|(A_1 \cup A_3) \cap M_2) = 1/2$. Thus $\boxed{\text{it does not matter if we switch the door.}}$

3. *Linearity of expectation.* We play the following game. A fair die is rolled n times. Every time we roll a number that differs by one from the number we got on our previous roll we win \$1. Every time we roll a number that is equal to the one we got on our previous roll we lose \$1. (For example, if we get numbers $1, 3, 3, 4, 6, 5, 3$ as a result of seven rolls, then we win \$1 in total, getting \$1 on the fourth and sixth roll, and losing \$1 on the third roll.) What are our expected winnings?

Solution: Let X_i be the random variable indicating the amount we win (or lose) on the ith roll. Then for $i \geq 2$, $p(X_i = -1) = 1/6$, $p(X_i = 1) = 2/6 * 1/6 + 4/6 * 2/6 = 5/18$, as the probability of winning one dollar is $1/6$ if the previous roll was 1 or 6, and $2/6$, otherwise. Thus $\mathrm{E}(X_i) = 5/18 - 1/6 = 1/9$. Let $X = X_2 + X_3 + \ldots + X_n$ be the random variable indicating our total winnings. Then,by linearity of expectation,

$$\mathrm{E}(X) = \sum_{i=2}^{n} \mathrm{E}(X_i) = \boxed{\frac{n-1}{9}.}$$

4. *Markov inequality.* Let X be a random variable taking only non-negative values, and let c be a positive constant. Show that

$$p(X \geq c) \leq \mathrm{E}(X)/c.$$

Solution: We have

$$\mathrm{E}(X) = \sum_{v} v \cdot p(X = v) \geq \sum_{v \geq c} v \cdot p(X = v) \geq \sum_{v \geq c} c \cdot p(X = v) = c \cdot p(X \geq c),$$

implying the desired inequality.

5. *Binomial distribution.* The Stanley Cup winner is determined in the final series between two teams. The first team to win 4 games wins the Cup. Suppose that Montréal Canadiens advance to the final series, and they have a probability of 0.6 to win each game, and the game results are independent of each other. Find the probability that

a) Canadiens win the Stanley cup.

b) Seven games are required to determine the winner.

Solution:
a): We may assume without loss of generality that the series consists of seven games and Canadiens win the cup if and only if they win at least 4 of these. This happens with probability.

$$\sum_{k=4}^{7} \binom{7}{k} (0.6)^k (0.4)^{7-k} = 35\frac{3^4 2^3}{5^7} + 21\frac{3^5 2^2}{5^7} + 7\frac{3^6 2}{5^7} + \frac{3^7}{5^7}$$
$$= \boxed{0.710208}$$

b): This event occurs if and only if each team wins three out of the first six games, which happens with probability

$$\binom{6}{3} (0.6)^3 (0.4)^3 = \boxed{0.27648}$$

6. *Geometric distribution.* Suppose we run repeated independent Bernoulli trials (with success probability p) until we obtain a success. Let the random variable X be the number of trials needed before we obtain a success.

a) Calculate the probability $P(X = k)$ for an integer k.

b) Prove that $\mathrm{E}(X) = 1/p$.

Solution:
a):

$$P(X = k) = p(1-p)^{k-1}.$$

(We assume that the number of trials includes the last successful one.)

b): Recall the formula for the sum of infinite geometric series:

$$\sum_{k=0}^{\infty}(1-p)^k = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

Using the formula, we have

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{k=1}^{\infty} kp(1-p)^{k-1} \\
&= \sum_{k=1}^{\infty} p(1-p)^{k-1} + \sum_{k=2}^{\infty}(k-1)p(1-p)^{k-1} \\
&= p \cdot \frac{1}{p} + (1-p)\sum_{l=1}^{\infty} lp(1-p)^{l-1},
\end{aligned}$$

where we substituted l for $k-1$ in the last summation. Note that the last sum is equal to $\mathrm{E}(X)$. Thus we have

$$\mathrm{E}(X) = 1 + (1-p)\mathrm{E}(X),$$

implying the desired result.