MATH 340: Discrete Structures II. Winter 2016.
Due in class on Tuesday, April 5th.

Assignment #5: Enumeration.

1. *Combinatorial identities.*

a) Give an algebraic proof of the following identity:

$$\binom{n+1}{m+1} = \sum_{k=m}^{n} \binom{k}{m}$$

b) Give a combinatorial (bijective) proof of the identity in a).

2. *Labelled trees.*
Let $f : [n] \to [n]$ be a function, and let T_f be a labelled tree on n vertices, constructed from f using the procedure demonstrated in class. Suppose that f takes exactly k different values. Show that T_f has at least $n-k$ and at most $n-k+2$ leaves.

3. *Catalan numbers.*
Give a bijection to show that the following is counted by Catalan numbers. The number of orderings of numbers $\{1, 2, \ldots, 2n\}$, such that

- the numbers $\{1, 3, \ldots, 2n-1\}$ appear in order,
- the numbers $\{2, 4, \ldots, 2n\}$ appear in order,
- $2k-1$ preceded $2k$ for every $1 \leq k \leq n$.

(For example, for $n = 2$ the orderings 1234 and 1324 are the only ones satisfying the above conditions. The ordering 3124 is invalid, as $\{1, 3\}$ is not in order, 1243 is invalid, as 4 precedes 3.)

4. *Plane trees.*
Show that for $n \geq 2$ there are exactly $\frac{1}{n}\binom{2n-2}{n-1}$ rooted plane trees on $n+1$ vertices in which the root vertex has degree two.

5. *Generating functions.*

For the following recurrences, find the ordinary generating function $F(x)$ and use it to obtain a closed formula for $f(n)$.

a) $f(n) = 3f(n-1) - 2f(n-2)$ for $n \geq 2$, $f(0) = 3$, $f(1) = 5$,

b) $f(n) = 8f(n-1) - 16f(n-2)$ for $n \geq 2$, $f(0) = 0$, $f(1) = 4$.