MATH 340: Discrete Structures II. Winter 2016.
Due in class on Tuesday, March 29th.

Assignment #4: Discrete Probability II.

1. *The Birthday problem.* Suppose that the birthdays of n people in the room are uniformly distributed among the 365 days of the year. Estimate how large should n be to guarantee that the probability of some two people sharing a birthday is at least 99/100.

2. *Random walks on graphs.* Consider a random walk on the complete graph on 5 vertices, starting at some vertex v. Let w be another vertex, and let p_n denote the probability that after n steps the random walk is at w. (In particular, $p_0 = 0$, $p_1 = 1/4$.)

a) Show that $p_n = (1 - p_{n-1})/4$ for $n \geq 1$.

b) Deduce that

$$p_n = \frac{1}{5}\left(1 - \left(-\frac{1}{4}\right)^n\right).$$

3. *Balls and bins.* Suppose that we randomly drop n^2 balls into n bins. Give an upper bound on the expectation of the maximum number of balls in any bin.

4. *Deviation below the mean.* Prove the following variant of the Chernoff bound for the deviation below the mean. Let X be the sum of independent Bernoulli random variables, and let $\mu = \mathrm{E}[X]$. Show that, if $0 < \delta < 1$, then

$$p(X \leq (1-\delta)\mu) \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu} \leq e^{-\mu\delta^2/2}$$

5. *Applying Chernoff bounds.* Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with expectation $\frac{1}{2}$. (That is $p(X_i = 0) = p(X_i = 1) = \frac{1}{2}$ for $1 \le i \le n$.) Let

$$X = X_1X_2 + X_2X_3 + X_3X_4 + \ldots + X_{n-1}X_n.$$

a) Find $E[X]$.

b) Show that $p(X > E[X] + 2\sqrt{n}) < 1/2$ for sufficiently large n.

6. *Quicksort.* Let $x_1, x_2, \ldots, x_n$ be a permutation of numbers $1, \ldots, n$ chosen uniformly at random.

a) Show that the probability that the numbers i and j, such that $1 \le i \le j \le n$, are compared to each other by the Quicksort algorithm is equal to

$$\frac{2}{j - i + 1}.$$

b) Deduce that the expected number of comparisons made by the Quicksort algorithm is equal to

$$2\sum_{k=1}^{n-1} \frac{n-k}{k+1}.$$