MATH 550: Combinatorics. Winter 2013.

Final exam.
Due electronically at snorin@math.mcgill.ca
by 5PM on Tuesday, April 30th.

1. Turan-type problems.

Consider 3-graphs $F_4 = \{abc, abd, acd\}$ and $F_5 = \{abc, abd, cde\}$. We say that a 3-graph G is *cancellative* if it contains neither F_4 nor F_5 as a subgraph.

a) Let G be a 3-*partite* 3-graph on n vertices. (That is there exists a partition (A_1, A_2, A_3) of $V(G)$ so that $|e \cap A_i| = 1$ for every $e \in G$ and every $i = 1, 2, 3$.) Show that G is cancellative.

Let G be a cancellative 3-graph. For every $x \in V(G)$ define a graph L_x on $V(G)$ so that $\{y, z\} \in L_x$ if and only if $\{x, y, z\} \in G$.

b) Show that for every $\{x, y, z\} \in G$ one has $L_x \cap L_y = \emptyset$.

c) Suppose that for some $\{x, y, z\} \in G$ the graph $L_x \cup L_y \cup L_z$ contains a K_4 subgraph with the vertex set $\{w_1, w_2, w_3, w_4\}$. Show that $L_x, L_y, L_z, L_{w_1}, L_{w_2}, L_{w_3}$ and L_{w_4} are pairwise disjoint.

d) Deduce that $|L_x| \leq (\frac{2}{9} + o(1))\binom{n}{2}$ for some $x \in V(G)$.

e) Conclude that $\pi(\{F_4, F_5\}) = \frac{2}{9}$.

2. Ramsey Numbers.

Let G be a graph with $V(G) = [17]$ and $x, y \in V(G)$ adjacent if and only if

$$(x - y) \bmod 17 \in \{\pm 1, \pm 2, \pm 4, \pm 8\}.$$

a) Show that neither G nor the complement of G contains a K_4 subgraph.

b) Deduce that $R(4, 4) = 18$.

3. Convexity & Ramsey theory.

Show that for all positive integers n, d there exists a positive integer N such that in every set of N points $P \subseteq \mathbb{R}^d$ in general position one can find a subset of size n in convex position. (A set of P is in *general position* in $\mathbb{R}^d$ if no set of $d + 1$ points in P is affinely dependent.)

4. Szemeredi-Trotter theorem.

Show that there exists a constant $C > 0$ so that for any finite set $P \subseteq \mathbb{R}^2$ one has

$$|\{(a,b) \mid a,b \in P,\ \|a-b\| = 1\}| \leq C|P|^{4/3}.$$

(*Hint:* Modify Szemeredi-Trotter theorem so that it applies to circles instead of lines.)

5. Combinatorial Nullstellensatz.

Let $(X, \mathcal{F})$ be a set-system such that $|\mathcal{F}| > 2|X|$. Show that there exists $\mathcal{F}' \subseteq \mathcal{F}, \mathcal{F}' \neq \emptyset$ such that for every $x \in X$ the number of elements of $\mathcal{F}'$ containing x is divisible by 3.

6. Maximum Cut.

Let G be a graph with $V(G) = [2n]^{(n)}$ – the set of all n element subsets of a $2n$ element set. Let $S, T \in V(G)$ be adjacent if and only if $|S \cap T| = 1$.

a) Show that if n is odd then

$$\frac{\text{MaxCut}(G)}{|E(G)|} \leq \frac{n-1}{n}.$$

b) Show that

$$\frac{\text{MaxCut}^{\circ}(G)}{|E(G)|} \geq \frac{n-1}{n}.$$