On the notion of flat 2-functors

(joint work with Emilia Descotte and Martin Szyld)

Let $(\mathcal{I},\ \Sigma)$ be a pair where $\mathcal{I}$ is a 2-category and Σ a distinguished 1-subcategory. A σ-cone for a 2-functor $\mathcal{I} \longrightarrow \mathcal{C}$ is a lax cone such that the 2-cells corresponding to the distinguished arrows are invertible, σ-limits are as usual the universal σ-cones. Similarly we define σ-dicones and σ-ends for 2-functors $\mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathcal{B}$.

The beautiful thing is that σ-ends can be expressed as σ-limits. This is an important fact of limits and ends in category theory that was not available in 2-dimensional category theory.

We introduce the notion of σ-filtered pair which generalises 2-filterness, and we develop a successful theory of flat 2-functors. We define a 2-functor $\mathcal{A} \xrightarrow{P} \mathcal{C}at$ to be *flat* when its left bi-Kan extension $\mathcal{H}om_s(\mathcal{A}^{op}, \mathcal{C}at) \xrightarrow{P^*} \mathcal{C}at$ along the Yoneda 2-functor $\mathcal{A} \xrightarrow{h} \mathcal{H}om_s(\mathcal{A}^{op}, \mathcal{C}at)$ is left exact (where $\mathcal{H}om_s(\mathcal{A}^{op}, \mathcal{C}at)$ denotes the 2-category of 2-functors, 2-natural transformations and modifications, and by left exact we understand preservation of finite weighted bilimits). The main result is:

A 2-functor $\mathcal{A} \xrightarrow{P} \mathcal{C}at$ is flat if and only if there is a σ-filtered pair $(\mathcal{I}^{op},\ \Sigma)$, a 2-diagram $\mathcal{I} \xrightarrow{X} \mathcal{A}$, and P is pseudo-equivalent to the σ-bicolimit of the composition $\mathcal{I}^{op} \xrightarrow{X} \mathcal{A}^{op} \xrightarrow{h} \mathcal{H}om_s(\mathcal{A}, \mathcal{C}at)$.

This establishes that the 2-category of *points* of the "2-topos" of 2-presheaves $\mathcal{H}om_s(\mathcal{A}^{op}, \mathcal{C}at)$ ($\mathcal{A}$ small), is equivalent to the 2-category σ-$\mathcal{P}ro(\mathcal{A})$ of σ-pro-objects of $\mathcal{A}$. In other words, the 2-category of *models* is equivalent to the 2-category σ-$\mathcal{I}nd(\mathcal{A}^{op})$ of σ-ind-objects of $\mathcal{A}^{op}$.