① Operation A_1

$$A_1 : [A_1, X] \longrightarrow X_3$$

where : $A_1 :=:$
$$\boxed{\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet \arrow[r] & \bullet \arrow[from=U, to=M, shorten=2pt] \arrow[from=M, to=D, shorten=2pt] \end{tikzcd}}$$
("distributivity")

For (a, b, f) such that

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] & \circ \arrow[r, "f"] & \bullet \arrow[from=U, to=M, shorten=2pt, "a"] \arrow[from=M, to=D, shorten=2pt, "b"] \end{tikzcd}$$

$A_1(a, b, f)$ is a 3-cell as follows.

Consider

$$\begin{tikzcd} a^0 \arrow[r, bend left=60, "a^1", ""{name=U, below}] \arrow[r, "a_1"', ""{name=M}] \arrow[r, bend right=60, "b_1"', ""{name=D, above}] & a_0 \arrow[r, "f"] & f_0 \arrow[from=U, to=M, shorten=2pt, "a"] \arrow[from=M, to=D, shorten=2pt, "b"] \end{tikzcd}$$

the non-identity elements | in the magma X

of the form $a^0 \Rrightarrow f_0$

generated are: (without the fillings Φ_1 ...):

$$\begin{tikzcd}
a'f \arrow[d, "af"'] \arrow[dd, bend left=60, "(ab)f"] \\
a_1f \arrow[d, "bf"'] \arrow[r, phantom, "\overset{A_1}{\Longrightarrow}"] & {} \\
b_1f
\end{tikzcd}$$

$A_1(a,b,f)$ is to be a 2-cell of the form

$$A_1(a,b,f) : (af)(bf) \longrightarrow (ab)f$$

② Similarly: operation A_2 for

$$A_2 :=: \boxed{\begin{tikzcd} \cdot \arrow[r, "f"'] & \cdot \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & \cdot \end{tikzcd}}$$

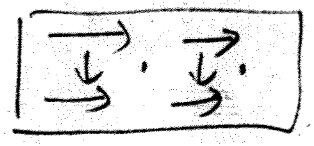
$$\begin{tikzcd}
f^0 \arrow[r, "f"'] & f_0 \arrow[r, shift left=4, "c'"] \arrow[r, "c_1"] \arrow[r, shift right=4, "d_1"'] & c_0
\end{tikzcd}
\quad : \quad
\begin{tikzcd}
fc' \arrow[d, "fc"'] \arrow[dd, bend left=60, "f(cd)"] \\
fc_1 \arrow[d, "fd"'] \arrow[r, phantom, "\overset{A_2(f,c,d)}{\Longrightarrow}"] & {} \\
fd_1
\end{tikzcd}$$

(with 2-cells $\downarrow c$ between c' and c_1, $\downarrow d$ between c_1 and d_1)

③ Let:

$$\begin{tikzcd} a^0 \arrow[r, bend left, "a'"] \arrow[r, bend right, "a_1"'] \arrow[r, phantom, "\downarrow a"] & a_0 \arrow[r, bend left, "b'"] \arrow[r, bend right, "b_1"'] \arrow[r, phantom, "\downarrow b"] & b_0 \end{tikzcd}$$

B-cell :



generate & define operation $(a, b) \longmapsto a ;_0 b$:

$$\begin{tikzcd} a'b' \arrow[r, "a'b"] \arrow[d, "ab'"'] & a'b_1 \arrow[d, "ab_1"] \arrow[dl, Rightarrow, "a ;_0 b"] \\ a_1b' \arrow[r, "a_1b"'] & a_1b_1 \end{tikzcd}$$

nodes : 1-cells

edges : 2-cells

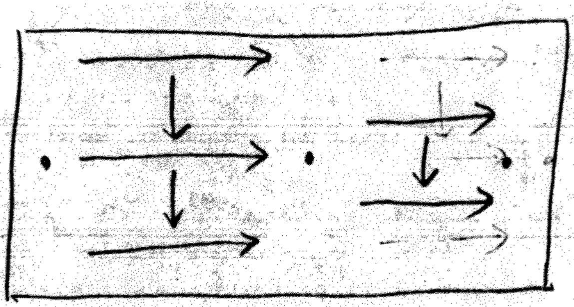
$$\underset{\substack{\uparrow \\ \text{3-cell}}}{\boxed{a ;_0 b}} : (a'b)(ab_1) \longrightarrow (ab')(a_1b)$$

'commutativity operation'

(Crans' operation)

⑤ Consider

A_3 : = :



Let:

$$\begin{tikzcd}[column sep=large]
a^0 \arrow[r, bend left=60, "a'"{name=U}] \arrow[r, "a_1"{name=M}] \arrow[r, bend right=60, "b_1"'{name=D}] & a_0 \arrow[r, bend left=40, "c'"{name=U2}] \arrow[r, bend right=40, "c_1"'{name=D2}] & c_0
\arrow[Rightarrow, from=U, to=M, "a"]
\arrow[Rightarrow, from=M, to=D, "b"]
\arrow[Rightarrow, from=U2, to=D2, "c"]
\end{tikzcd}$$

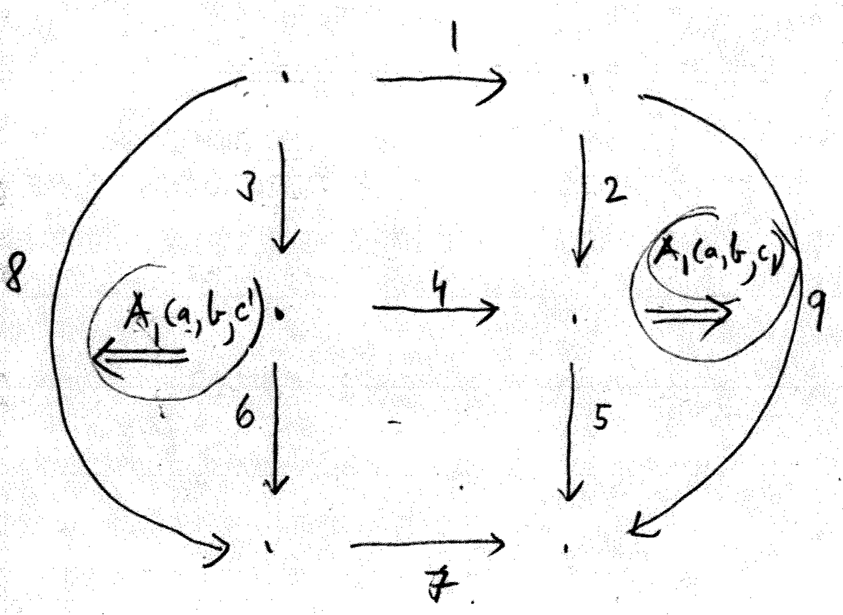
an is instantiation $\Phi : B \to X$

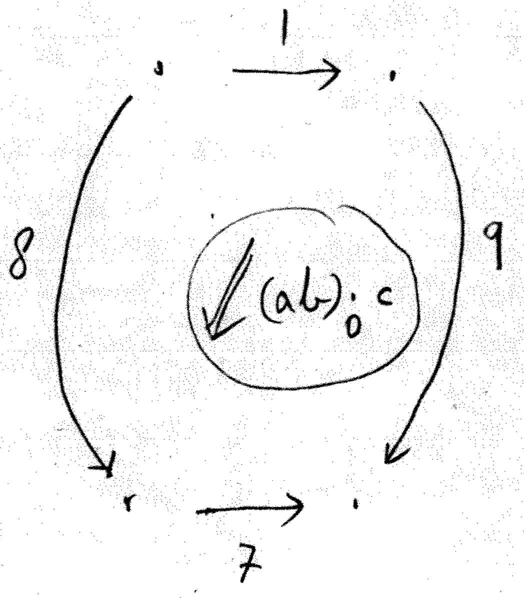
<u>Assume full associativity in X.</u>

Generated 1- and 2-cells:

$$\begin{tikzcd}
a'c' \arrow[r] \arrow[d, "ac'"'] \arrow[dd, bend right=70, "(ab)c'"'] & a'c_1 \arrow[d, "ac_1"] \arrow[dd, bend left=70, "(ab)c_1"] \\
a_1c' \arrow[r] \arrow[d, "bc'"'] & a_1c_1 \arrow[d] \\
b_1c' \arrow[r, "b_1c"'] & b_1c_1
\end{tikzcd}$$

abbreviate:





making:

$$\begin{tikzcd}
 & 3\cdot 4\cdot 5 \arrow[r, "3\cdot(b\underset{0}{;}c)"] & 3\cdot 6\cdot 7 \arrow[dr, "A_1(a,b,c')\cdot 7"] & \\
1\cdot 2\cdot 5 \arrow[ur, "(a\underset{0}{;}c)\cdot 5"] \arrow[dr, "1\cdot A_1(a,b,c_1)"'] & & \Downarrow A_3(a,b,c & 8\cdot 7 \\
 & 1\cdot 9 \arrow[urr, "(ab)\underset{0}{;}c"'] & &
\end{tikzcd}$$

nodes: 2-cells

edges: 3-cells

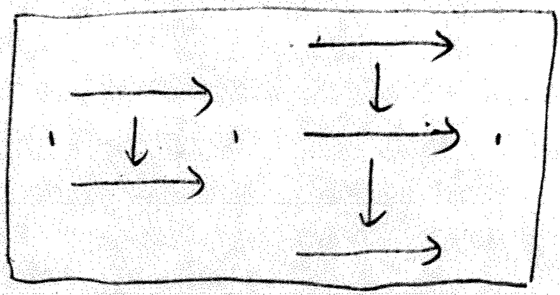
filling: 4-cell

the new operation

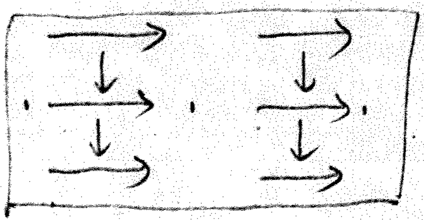
There is a symmetric treatment

for the arity

$(A_4$: =:

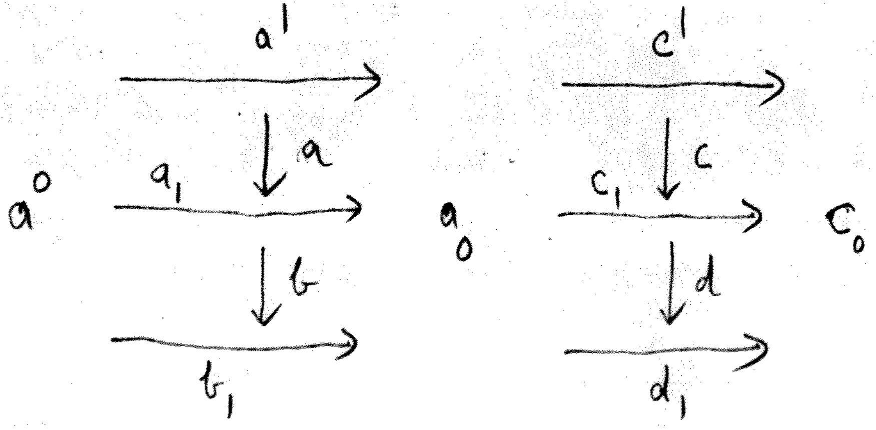


⑥ Arity:



Will give rise to an operation whose value is a 5-cell

Let:



generate:

1-cells & 2-cells:

$$\begin{tikzcd}[column sep=huge, row sep=large]
a'c' \arrow[r, "a'c"'] \arrow[d, "ac'"] \arrow[dd, bend right=60, "(ab)c'"'] \arrow[rr, bend left=40, "a'(cd)"] & a'c_1 \arrow[r, "a'd"'] \arrow[d, "ac_1"'] \arrow[dd, dashed, bend left, "(ab)c_1"] & a'd_1 \arrow[d, "ad_1"'] \arrow[dd, bend left=60, "(ab)d_1"] \\
a_1c' \arrow[r, "a_1c"] \arrow[d, "bc'"] \arrow[rr, dashed, bend right=20, "a_1(cd)"'] & a_1c_1 \arrow[r, "a_1d"] \arrow[d, "bc_1"'] & a_1d_1 \arrow[d, "bd_1"'] \\
b_1c' \arrow[r, "b_1c"'] \arrow[rr, bend right=40] & b_1c_1 \arrow[r] & b_1d_1
\end{tikzcd}$$

Abbreviate, with some 3-cells:
(eight of them here)

$$\begin{tikzcd}[column sep=huge, row sep=large]
\cdot \arrow[r, "1"] \arrow[rr, bend left=40, "3"] \arrow[rr, phantom, bend left=20, "\Uparrow"] \arrow[d, "4"] \arrow[dd, bend right=60, "6"'] \arrow[dd, phantom, bend right=30, "\Leftarrow"] \arrow[dr, phantom, "a_0;c \ \Downarrow"] & \cdot \arrow[r, "2"] \arrow[d, "10"'] \arrow[dd, dashed, bend left, "11"] \arrow[dr, phantom, "a_0;d"] & \cdot \arrow[d, "12"] \arrow[dd, bend left=60, "18"] \arrow[dd, phantom, bend left=30, "\Rightarrow"] \\
\cdot \arrow[r, "7"] \arrow[d, "5"] \arrow[rr, dashed, bend right=20, "8"] \arrow[dr, phantom, "b_0;c \ \Downarrow"] & \cdot \arrow[r, "9"] \arrow[d, "13"'] \arrow[dr, phantom, "b_0;d"] & \cdot \arrow[d, "14"'] \\
\cdot \arrow[r, "15"'] \arrow[rr, bend right=40, "17"'] \arrow[rr, phantom, bend right=20, "\Downarrow"] & \cdot \arrow[r, "16"'] & \cdot
\end{tikzcd}$$

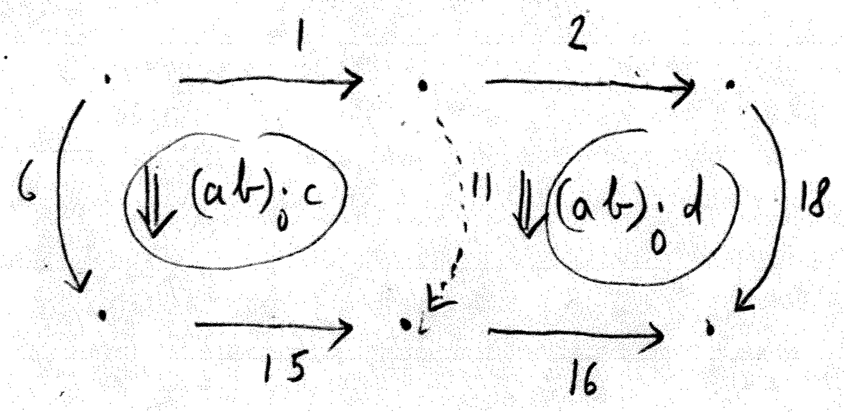
$A_2(a',c,d)$

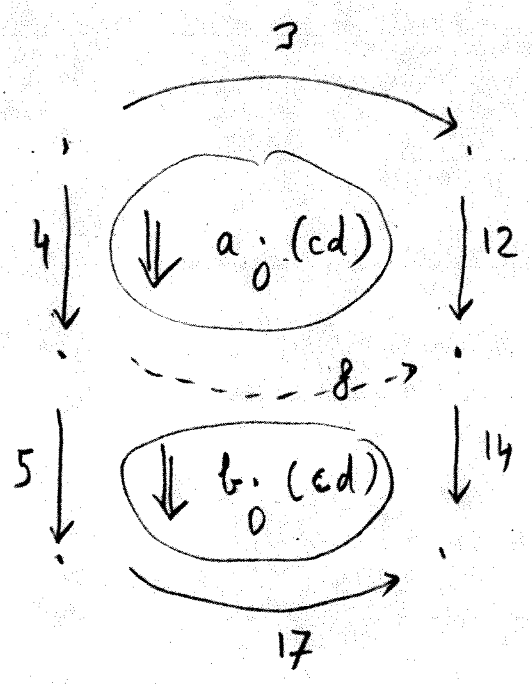
$A_1(a,b,d_1)$

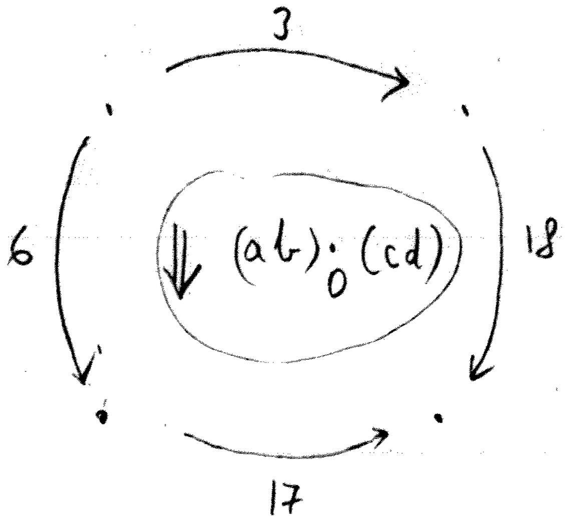
$A_1(a,b,c')$

$A_2(b_1,c,d)$

Five more 3-cells:



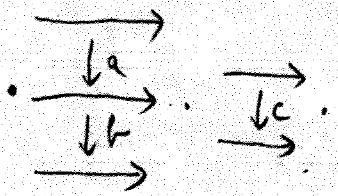


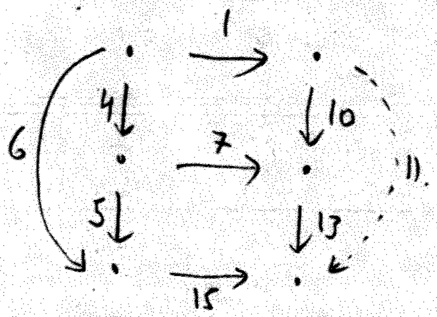


end of directly generated 3-cells (14 altogether)

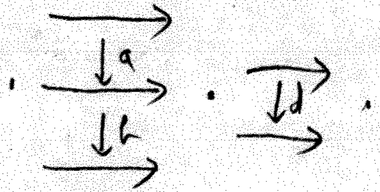
4-cells, $\bigcirc$ nine of them:

$\boxed{1}$ $\boxed{A_3(a,b,c)}$:

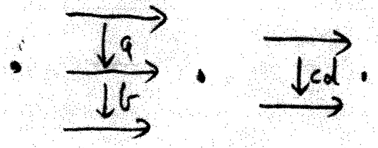




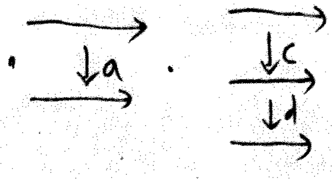
$\boxed{2}$ $\boxed{A_3(a,b,d)}$:



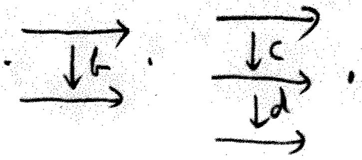
$\boxed{3}$ $\boxed{A_3(a,b,cd)}$:



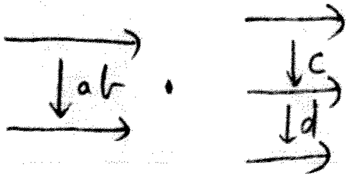
$\boxed{4}$ $\boxed{A_4(a,c,d)}$:



$\boxed{5}$ $\boxed{A_4(b,c,d)}$:



$\boxed{6}$ $\boxed{A_4(ab,c,d)}$:



$$\boxed{7}\ \boxed{\overbrace{A_2(a',c,d)}^{\textcircled{1}} \underset{1}{\cdot} \overset{\textcircled{2}}{A_1(a,b,d_1)}}\ :$$

$$\begin{tikzcd}
\bullet \arrow[r, "1"'] \arrow[rr, bend left, "3"] \arrow[rr, phantom, bend left=20, "\Uparrow \textcircled{1}"] & \bullet \arrow[r, "2"'] & \bullet \arrow[d, "12"'] \arrow[dd, bend left=60, "18"] \arrow[dd, phantom, bend left=25, "\Rightarrow \textcircled{2}"] \\
 & & \bullet \arrow[d, "14"'] \\
 & & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
 & \bullet \arrow[dr, "2"] \arrow[d, phantom, "\Downarrow \textcircled{1}"] & & \bullet \arrow[dr, "14"] \arrow[d, phantom, "\textcircled{2} \Downarrow"] & \\
\bullet \arrow[ur, "1"] \arrow[rr, bend right, "3"'] & {} & \bullet \arrow[ur, "12"] \arrow[rr, bend right, "18"'] & {} & \bullet
\end{tikzcd}$$

dimension = 1

$$\boxed{8}\quad \boxed{A_1(a,b,c') \underset{1}{\cdot} A_2(b_1,c,d)}$$

$$\begin{tikzcd}
\bullet \arrow[d] \arrow[dd, bend right=60] \arrow[dd, phantom, bend right=30, "\Leftarrow"] & & \\
\bullet \arrow[d] & & \\
\bullet \arrow[r] \arrow[rr, bend right=50] \arrow[rr, phantom, bend right=25, "\Downarrow"] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

$\boxed{9}$ $\boxed{(a \underset{0}{;} c) \underset{1}{;} (b \underset{0}{;} d)}$:

$$\begin{tikzcd}
\bullet \arrow[r, "1"] \arrow[d, "4"'] \arrow[dr, phantom, "(a \underset{0}{;} c)\Downarrow"] & \bullet \arrow[d, "10"] & \\
\bullet \arrow[r, "7"'] & \bullet \arrow[r, "9"] \arrow[d, "13"'] \arrow[dr, phantom, "\Downarrow (b \underset{0}{;} d)"] & \bullet \arrow[d, "14"] \\
 & \bullet \arrow[r, "16"'] & \bullet
\end{tikzcd}$$

⇙ part of diagram on T_2 7

(end of list of directly generated 4-cells)

"Long" composite 2-cells:

of the form $a^1c^1 \longrightarrow b_1d_1$

1	$1\cdot 2\cdot 12\cdot 14$
2	$1\cdot 2\cdot 18$
3	$1\cdot 10\cdot 9\cdot 14$
4	$1\cdot 10\cdot 13\cdot 16$
5	$1\cdot 11\cdot 16$
6	$4\cdot 7\cdot 9\cdot 14$
7	$4\cdot 7\cdot 13\cdot 16$
8	$4\cdot 8\cdot 14$
9	$4\cdot 5\cdot 15\cdot 16$
10	$4\cdot 5\cdot 17$
11	$3\cdot 12\cdot 14$
12	$3\cdot 18$
13	$6\cdot 15\cdot 16$
14	$6\cdot 17$

T_2 13

T_2 13

$$\begin{tikzcd}
 & & & 7 \arrow[dr, "b\,;_0\,c"] \arrow[ddd, phantom, "\boxed{11}\ \boxed{A_3(a,b,c)}\ \Downarrow"] & & & \\
 & & 4 \arrow[ur, "a\,;_0\,c"] \arrow[ddr, "A_1(a,b,c_1)"] & & 9 \arrow[dr, "A_2(b_1,c,d)"] \arrow[ddd, "A_1(a,b,c')"] \arrow[ddddrr, phantom, "\boxed{8}\ \Downarrow"] & & \\
 & 3 \arrow[ur, "b\,;_0\,d"] \arrow[ddr, phantom, "\Downarrow\ \boxed{2}\ \boxed{A_3(a,b,d)}"] & & & & 10 \arrow[dddr, "A_1(a,b,c')"] & \\
 & & & 5 \arrow[dr, "ab\,;_0\,c"] & & & \\
 & & 2 \arrow[ur, "ab\,;_0\,d"'] \arrow[ddrr, "A_2(a',c,d)"] & & 13 \arrow[drr, "A_2(b_1,c,d)"'] \arrow[dd, phantom, "\boxed{6}\ \boxed{A_4(ab,c,d)}\ \Downarrow"] & & \\
1 \arrow[uuur, "a\,;_0\,d"] \arrow[urr, "A_1(a,b,d_1)"'] \arrow[drr, "A_2(a',c,d)"'] \arrow[rr, phantom, "\Downarrow\ \boxed{7}"] & & & & & & 14 \\
 & & 11 \arrow[rr, "A_1(a,b,d_1)"'] & & 12 \arrow[urr, "(ab)\,;_0\,(cd)"'] & &
\end{tikzcd}$$

T_2 14

$$\begin{tikzcd}
 & & & & 7 \arrow[drr] & & & & \\
 & & 4 \arrow[urr] & & & & 9 \arrow[dr] & & \\
 & 3 \arrow[ur] \arrow[drr, "a\,;_0\,c"'] & & & & & & 10 \arrow[ddr] & \\
 & & & 6 \arrow[uuur, "b\,;_0\,d"] \arrow[rr, "A_2(a_1,c,d)"'] & & 8 \arrow[urr, "b\,;_0\,cd"'] & & & \\
 1 \arrow[uur] \arrow[drrr] & & & & & & & & 14 \\
 & & & 11 \arrow[uurr, "a\,;_0\,cd"'] \arrow[rr] & & 12 \arrow[urrr] & & &
\arrow[from=2-3, to=4-4, phantom, "\boxed{9}\ (a\,;_0\,c)\,;_1\,(b\,;_0\,d)\ \Downarrow"]
\arrow[from=1-5, to=4-6, phantom, "\boxed{5}\ A_4(b,c,d)\ \Downarrow"]
\arrow[from=3-2, to=6-4, phantom, "\boxed{4}\ A_4(a,c,d)\ \Downarrow"]
\arrow[from=3-8, to=6-6, phantom, "\boxed{3}\ A_3(a,b,cd)\ \Downarrow"]
\end{tikzcd}$$

Pasting of diagram T_2 13 :

$$1 - 3 - 4 - 7 - \underline{9 - 10 - 14}$$

$$\downarrow \boxed{8}$$

$$1 - 3 - \underline{4 - 7 - \overline{9 - 13}}\overline{- 14}$$

$$\downarrow \boxed{1}$$

$$\underline{1 - 3 - \overline{4 - 5}}\overline{- 13} - 14$$

$$\downarrow \boxed{2}$$

$$\overline{1 - \underline{2}}\underline{- 5 - 13 - 14}$$

$$\downarrow \boxed{6}$$

$$\underline{1 - \overline{2 - 12}}\overline{- 14}$$

$$\downarrow \boxed{7}$$

$$\overline{1 - 11 - 12} - 14$$

$$\begin{tikzcd}
① \arrow[d, "\boxed{8}"] \\
2 \arrow[d, "\boxed{1}"] \\
3 \arrow[d, "\boxed{2}"] \\
4 \arrow[d, "\boxed{6}"] \\
5 \arrow[d, "\boxed{7}"] \\
⑥
\end{tikzcd}$$

4-cells

(whiskerings omitted)

Pasting of diagram T_2 14 :

$$\begin{tikzcd}
1 - \underline{3 - 4 - 7} - 9 - 10 - 14 \arrow[d, "\boxed{9}"'] & ① \text{ (same as before)} \arrow[d, "\boxed{9}"'] \\
1 - \overline{3 - {}}\overline{\underline{6 - 7}}\underline{{} - 9 - 10} - 14 \arrow[d, "\boxed{5}"'] & 7 \arrow[d, "\boxed{5}"'] \\
\underline{1 - 3 - \overline{6 - 8}} - 10 - 14 \arrow[d, "\boxed{4}"'] & 8 \arrow[d, "\boxed{4}"'] \\
\overline{1 - {}}\overline{\underline{11}}\underline{{} - 8 - 10 - 14} \arrow[d, "\boxed{3}"'] & 9 \arrow[d, "\boxed{3}"'] \\
1 - \overline{11 - 12 - 14} & ⑥
\end{tikzcd}$$

same as before

Finally:

$$\begin{tikzcd}
 & & 3 \arrow[r, "\boxed{2}"] & 4 \arrow[dr, "\boxed{6}"] & & \\
 & 2 \arrow[ur, "\boxed{1}"] & & & 5 \arrow[dr, "\boxed{7}"] & \\
① \arrow[ur, "\boxed{8}"] \arrow[dr, "\boxed{9}"'] & & & \Downarrow A_5(a, b, c, d) & & ⑥ \\
 & 7 \arrow[dr, "\boxed{5}"'] & & & 9 \arrow[ur, "\boxed{3}"'] & \\
 & & 8 \arrow[urr, "\boxed{4}"'] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & 3 \arrow[rr, "A_3(abd)"] & & 4 \arrow[dr, "A_4((ab)cd)"] & & \\
 & & 2 \arrow[ur, "A_3(abc)"] & & & & 5 \arrow[dr, "A_2(a'cd)\cdot_1 A_1(abd_1"] & \\
① \arrow[urr, "A_1(abc')\cdot_1 A(b_1cd)"] \arrow[drr, "(a\cdot_0 c)\cdot_1(b\cdot_0 d)"'] & & & & \Downarrow\ A_5\left(\begin{array}{ccc} \longrightarrow & & \longrightarrow \\ \downarrow a & \cdots & \downarrow c \\ \longrightarrow & & \longrightarrow \\ \downarrow b & \cdots & \downarrow d \\ \longrightarrow & & \longrightarrow \end{array}\right) & & & ⑥ \\
 & & 7 \arrow[drr, "A_4(bcd)"'] & & & & 9 \arrow[ur, "A_3(ab(cd))"'] & \\
 & & & & 8 \arrow[urr, "A_4(acd)"'] & & &
\end{tikzcd}$$