Notes for the would-have-been talk of 2019 Nov 26

The notes continue those for Oct 08 and November 12.
The present notes give details of the proofs for
the classical Quillen model structure, as well as for
the formulation of the model structure, due to André Joyal,
whose fibrant objects are the quasi-categories.
The present notes only start giving the proofs; roughly,
a set again as much as these notes will contain the
completion. The present notes are detailed, in fact,
maybe overly detailed; I expect that they can be understood
without knowing anything of the (vast!) existing
literature: I am giving an elementary proof. Of
course, the Oct 08 & Nov 12 notes are a necessary

Background.

The present notes are not organized as a final write-up; a more aesthetic order is obviously called for a future write-up. For instance, it is reasonable to start the reading with Lemma 6 on page $\boxed{\boxed{23}}$. The proof of Lemma 6 is not complete as it is — but only in an inessential manner. The proof involving 1-simplices is, I think, sufficiently suggestive for what happens for n-simplices for $n \geq 2$. Of course, this situation is only temporary: I am giving out these notes as compensation for my failure of not giving the announced talk Nov 26.

2019 Nov 22

The simplicial set H(X) of internal homotopies of the simplicial set X

The simplicial set associated with a poset

$\boxed{1\odot}$ Poset : the category of partially ordered sets $(P, \leq)$

Δ : the (simplicial) category of abstract simplicies

Δ is a full subcategory of Poset; inclusion $i : \Delta \to \mathrm{Poset}$

Every poset P gives rise to a simplicial set $\underbrace{}_{}$(sset) $\Delta[P] \in \mathrm{Set}^{\Delta^{op}}$ $(= \mathrm{sSet})$:

Define functor $\Delta[-] : \mathrm{Poset} \longrightarrow \mathrm{Set}^{\Delta^{op}}$ as the composite functorially

$$\begin{tikzcd}
\mathrm{Poset} \arrow[r, "\text{yoneda emb.}"'] & \mathrm{Set}^{\mathrm{Poset}^{op}} \arrow[r, "{\mathrm{Set}^{i^{op}} = (-)\circ i^{op}}"'] & \mathrm{Set}^{\Delta^{op}}
\end{tikzcd}$$

Details: (i) $\boxed{1.1(1i)}$: For $P \in \mathrm{Poset}$, $\Delta[P]$ is the sset for which

$$\boxed{\underset{\Delta[P]([n])}{\underset{\shortparallel}{\Delta[P]_n}} = \mathrm{Poset}(\underset{i[n]}{\underset{\shortparallel}{[n]}}, P)}$$

In other words, an n-simplex in $\Delta[P]$ is an order-preserving map $[n] \longrightarrow P$.

Notation: ~~Definition~~ Let $X \in sSet = Set^{\Delta^{op}}$, $k \in \mathbb{N}$, $x \in X_n$; $f : [k] \longrightarrow [n]$ in Δ

Write $X_f = X(f) : X_n \to X_k$; $\boxed{xf \underset{\text{def}}{=} x^*(f) \underset{\text{def}}{=} X(f)(x)}$ This abbreviation is reasonable since $x^*(fg) = (x^*f)^*g = xfg$. In the case $X = \Delta[P]$,

$[l] \xrightarrow{g} [k] \xrightarrow{f} [n]$

$x \in X_n$, $x : [n] \longrightarrow P$, $f : [k] \longrightarrow [n]$, we get

$\boxed{x^*f = x \circ f}$ ($= xf$ simply)

Facts (simple): $\boxed{(i)}$ the non-degenerate n-simplices of $\Delta[P]$ are the injective $f : [n] \longrightarrow P$

$\boxed{(ii)}$ Let $\text{Lin}_n(P)$ be the set of subsets $s \subseteq P$ that are of cardinality $n+1$ and which are linearly ordered by the order $\leq$ of P. When $f \in \Delta[P]_n^{\text{nondeg}}$ ($f : [n] \rightarrowtail P$ injective), $\text{Im}(f) = \{ f(i) \mid i \in [n] \} \subseteq$ belongs to $\text{Lin}_n(P)$; the mapping $f \longmapsto \text{Im}(f)$ is a bijective mapping

$$\Delta[P]_n^{\text{nondeg}} \xrightarrow{\cong} \text{Lin}_n(P) \text{ ; inverse: } s \longmapsto \vec{s}.$$

~~For $s \in \text{Lin}_n(P) = \{ [illegible]$ [illegible]~~

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Terminology:

$\boxed{\text{(iii)}}$ A *k-face* of $x \in X_n$ is any $x^* f \in X_k$ for an *injective* $f : [k] \to [n]$ (so, x itself is considered here to be a face of x, the unique n-face of x).

For $X = \Delta[P]$, $s \in \mathrm{Lin}_n(P)$, the k-faces of $\hat{s} \in \Delta[P]_n$ (

all $\hat{t} \in \Delta[P]_k$, for $t \subseteq s$, $\# t = k+1$.

$\boxed{\text{(iv)}}$ Let $\mathrm{Lin}_{\neq \emptyset}(P) = \bigcup_{n \in \mathbb{N}} L_n(P)$. For a subcomplex $Y \subseteq \Delta[P]$ the set $\{ s \in \mathrm{Lin}_{\neq 0}(P) \mid \hat{s} \in |Y| \}$ is a *down closed* subset S of $\mathrm{Lin}_{\neq 0}(P)$: $s \in S$, $t \subseteq s$, $t \neq \emptyset \Rightarrow t \in S$. Conversely, for each down-closed subset S of $\mathrm{Lin}_{\neq \emptyset}(P)$, there is a unique subcomplex $Y \subseteq X$ such that

$$\hat{s} \in Y \iff s \in S \quad (s \in \mathrm{Lin}_{\neq 0}(P)).$$

Let, write $\hat{S}$ for this $Y \subseteq X$. $\hat{S}$ is the unique subcomplex Z of X for which $\{ \hat{s} \mid s \in S \} \subseteq |Z|$, and $|Z| - \{ \hat{s} \mid s \in S \}$ consists of degeneracies only.

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$\boxed{2}$ The significant examples.

For each n, $[n] \in \Delta \subseteq$ Poset. Let $\llbracket n \rrbracket = [1] \times [n]$, product in Poset.

As a set, $\llbracket n \rrbracket = \{(0,k) : k \in [n]\} \,\dot\cup\, \{(1,k) : k \in [n]\}$

Here, we write k for $(0,k)$, and $\underline{k}$ for $(1,k)$. The order on $\llbracket n \rrbracket$:

for $i, j \in [1]$, $k, l \in [n]$, $(i,k) \le (j,l) \iff i \le j$ & $k \le l$.

This means that for k, l in $[n]$:

in $\llbracket n \rrbracket$: $k \le l \iff k \le \underline{l} \iff \underline{k} \le \underline{l} \iff k \le l$ in $[n]$

and $\underline{k} \not\le l$: never.

For $n=0$: $\llbracket 0 \rrbracket = \{0 \to \underline{0}\}$

$\llbracket 1 \rrbracket = \{0, 1, \underline{0}, \underline{1}\}$

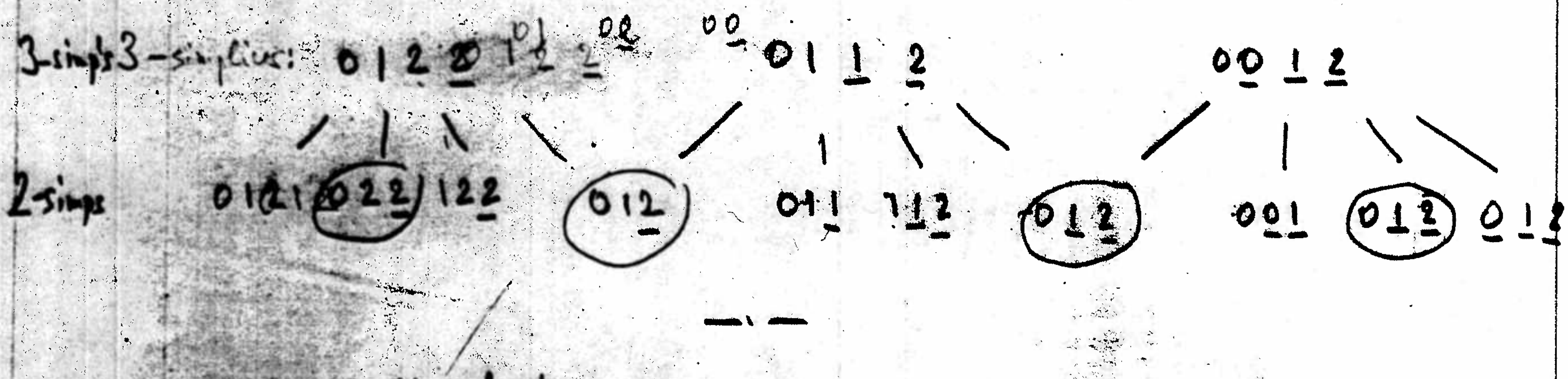
$$\begin{tikzcd}
1 \arrow[r] & \underline{1} \\
0 \arrow[r] \arrow[u, bend left] & \underline{0} \arrow[u, bend right]
\end{tikzcd}$$

$\llbracket 2 \rrbracket =$

$$\begin{tikzcd}
2 \arrow[r] & \underline{2} \\
1 \arrow[r] \arrow[u, bend left] & \underline{1} \arrow[u, bend right] \\
0 \arrow[r] \arrow[u, bend left] & \underline{0} \arrow[u, bend right]
\end{tikzcd}$$

Nov22 $\boxed{5}$

~~Th~~The non-degenerate 1-simplices of $[\![2]\!]$:



We have the functor

$$[\![-]\!] = [1] \times [-] : \Delta \to \text{Poset}$$

$$n \quad \begin{tikzcd} {[n]} \arrow[d, "\alpha"'] \\ {[m]} \end{tikzcd} \longmapsto [1] \times \alpha \quad \begin{tikzcd} {[\![n]\!]} \arrow[d] \\ {[\![m]\!]} \end{tikzcd} = [\![\alpha]\!]$$

Nov 22-29

$\boxed{5.1}$

<u>Remark</u> We will consider the sset

$$\Delta[n] \underset{\text{def}}{=} \left(\Delta[-] \circ [-]\right)[n]$$

(composite)

the functor $\Delta \xrightarrow[{[-]}]{} \text{Poset} \longrightarrow \text{Set}^{\Delta^{op}}$ applied to $[n] \in \Delta$

This is nothing new; it is the same thing as the product

$$\Delta[1] \times \Delta[n] \quad \text{in} \quad \text{Set}^{\Delta^{op}}$$

[I just prefer the more "geometric" view of $\Delta[n]$ as given above; The 'geometry' being commented on above.]

of $\Delta[P]$ for any poset P is

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$\boxed{\boxed{5.2}}$

$\boxed{3}$ **The complex of homotopies**

Def. Given the $X \in \mathrm{Set}^{\Delta^{op}}$, define $H(X) \in \mathrm{Set}^{\Delta^{op}}$ as the composite

$$\begin{tikzcd}
\Delta^{op} \arrow[r, "{[-]^{op}}", "{\underset{\text{see}}{[\uparrow]^{op}}\ \boxed{2}}"'] & \mathrm{Poset}^{op} \arrow[r, "{\Delta[-]^{op}}", "{\Delta\underset{\text{see}}{[\uparrow]^{op}}\ \boxed{1}}"'] & \mathrm{Set}^{\Delta^{op}} \arrow[r, "{\mathrm{Set}^{\Delta^{op}}(-, X)}"] & \mathrm{Set}
\end{tikzcd}$$

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In short

$$H(X) = \mathrm{Set}^{\Delta^{op}}\left(\Delta(\underset{\Delta}{[\![-]\!]}), X\right)$$

$$H(X)_n = \mathrm{Set}^{\Delta^{op}}\left(\Delta([\![n]\!]), X\right)$$

An n-simplex of $H(X)$ is a simplicial map

$$x : \Delta([\![n]\!]) \longrightarrow X.$$

For $a : [m] \to [n]$ in Δ, the action of a:

$$H(X)_a : H(X)_n \longrightarrow H(X)_m$$

$$\Delta([\![n]\!]) \xrightarrow{x} X \longmapsto \underbrace{\Delta([\![m]\!]) \xrightarrow{\Delta[\![a]\!]} \Delta([\![n]\!]) \xrightarrow{x} X}$$

$$\boxed{H(X)_a(x) = x \circ \Delta([\![a]\!])}$$

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Remarks: The above is re-stating the familiar concept of homotopy within a simplicial set.

Let $n \in \mathbb{N}$. Denote by $j_0, j_1 : [n] \longrightarrow \llbracket n \rrbracket$ the Poset-maps

$$i \overset{j_0}{\longmapsto} (0, i)$$

$$i \overset{j_1}{\longmapsto} (1, i)$$

Using the functor $\Delta[-] : \text{Poset} \longrightarrow \text{Set}^{\Delta^{op}}$, take any

$$x : \Delta\llbracket n \rrbracket \longrightarrow X$$

an "n-homotopy", and form

$$\begin{tikzcd}
\Delta[n] \arrow[r, shift left, "{\Delta[j_0]}"] \arrow[r, shift right, "{\Delta[j_1]}"'] \arrow[rr, bend left=50, "{x_0 = x \circ \Delta[j_0]}"] \arrow[rr, bend right=50, "{x_1 = x \circ \Delta[j_1]}"'] & \Delta\llbracket n \rrbracket \arrow[r, "x"] & X
\end{tikzcd}$$

We say – classically – that x is a homotopy of x_0 & x_1;

$$x : x_0 \sim x_1$$

<u>NB1</u>. In the context of quasi-categories, we will require additionally that

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$\boxed{6.2}$

the 1-simplices, components of x, with

$$q_k : [1] \longrightarrow [n]$$
$$0 \longmapsto (0,k)$$
$$1 \longmapsto (1,k)$$

the 1-simplices $x(q_k) \in X_1$ $(k \in [n])$ be all invertible.

For any $x \in X_n$, we have the identity homotopy

$$\mathbb{1}_x : x \xrightarrow{\sim} x$$

defined by $$\Delta[n] \xrightarrow[\Delta[\pi_1]]{} \Delta[n] \xrightarrow{\hat{x}} X$$

where π_1: is the projection $\pi_1 : \underset{\substack{\| \\ [1]\times[n]}}{[n]} \longrightarrow n$

and $\hat{x}$: is the 'Yoneda-given' natural transformation

(for which $\hat{x}_{[n]}(1_{[n]}) = x$).

Nov22 - Nov23

$H(-)$ as a functor: H is the composite

$$\begin{tikzcd} \mathrm{Set}^{\Delta^{op}} \arrow[r, "\text{yoneda}"'] & \mathrm{Set}^{(\mathrm{Set}^{\Delta^{op}})^{op}} \arrow[r, "\mathrm{Set}^{\Delta[-]^{op}}"'] & \mathrm{Set}^{\mathrm{Poset}^{op}} \arrow[r, "\mathrm{Set}^{[-]^{op}}"'] & \mathrm{Set}^{\Delta^{op}} \end{tikzcd}$$

compositions

$$X \longmapsto \mathrm{Set}^{\Delta^{op}}(-, X) \longmapsto \mathrm{Set}^{\mathrm{Poset}^{op}}(\Delta[-], X) \longmapsto \mathrm{Set}^{\Delta^{op}}(\Delta[[-]], X)$$

$$\begin{tikzcd} X \arrow[d, "f"'] & H(X) = \mathrm{Set}^{\Delta^{op}}(\Delta[[-]] \xrightarrow{\ } X, X) \arrow[d] \\ Y & H(Y) = \mathrm{Set}^{\Delta^{op}}(\Delta[[-]], Y) \end{tikzcd}$$

$\varphi \circ (-) = H(\varphi)$

$$H(X) \xrightarrow{H(f)} H(Y)$$

$$H(X)_n \xrightarrow{H(f)_n} H(Y)_n$$

$$\boxed{H(\varphi)_n (x) = f \circ x}$$

$$\Delta([n]) \xrightarrow{x} X \longmapsto \Delta([n] \xrightarrow{x} X \xrightarrow{f} Y)$$

(Pirzkonat) Nov 24

$\boxed{4}$ The restricted homotopy complex

Situation: in the category sSet $=$ Set$^{\Delta^{op}}$, given

$$\begin{tikzcd} X \arrow[d, "f"] \\ U \arrow[u, bend left, "r"] \end{tikzcd} \qquad \text{condition: } frf = f \quad (\text{special case: } rf = \mathrm{id}_U)$$

Define subcomplex $H(X/U)$ of $H(X)$:

$n \in \mathbb{N}$: $\qquad \underbrace{H(X/U)_n} \subseteq H(X)_n$

An element $[[n]] \xrightarrow{x} X$ of $H.(X)_n$ is in $H(X/U)_n$

iff

two conditions, 1) and 2), are satisfied

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<u>Condition 1)</u>: this refers to f alone; r is not involved.

Using H as a functor, at the dimension n:

$$H(f)_n : H(X)_n \longrightarrow H(U)_n$$

maps $[n] \xrightarrow{x} X \longmapsto [n] \xrightarrow{x} X \xrightarrow{f} U$

$$x \longmapsto fx$$

Let: $[n] \xrightarrow{x} X$ =

Denote : $j_0, j_1 : [n] \longrightarrow [1 n]$:

$$j_0(i) = (0, i)$$
$$j_1(i) = (1, i)$$

$\pi_0, \pi_1 : [n] \longrightarrow [n]$: the projection $[1] \times [n] \longrightarrow [n]$

There are morphisms in Poset.

$x \in H(X/U)_n \underset{def}{\Longleftrightarrow}$ ... next page

$x \in H(X/U)_n \underset{def}{\Longleftrightarrow}$ the diagram in $Set^{\Delta^{op}}$ commutes; following

$$\begin{tikzcd}
\Delta[n] \arrow[r, "\Delta[\pi_2]"] \arrow[dr, "fx"'] & \Delta[n] \arrow[r, "\Delta[j_0]"] \arrow[d, phantom, "///"] & \Delta[n] \arrow[dl, "fx"] \\
& U &
\end{tikzcd}$$

$$fx = fx \circ \Delta[j_0] \circ \Delta[\pi_1]$$

(the functor $\Delta[-] : Poset \to Set^{\Delta^{op}}$ was applied twice)

<u>Explanations:</u> Condition 1) says that the $H(f)$-image of the homotopy $x : x_0 \sim x_1$ is the identity homotopy

$$\mathbb{1}_{f(x_0)} : f(x_0) \sim f(x_1)$$

(including the condition that $f(x_1) = f(x_0)$)

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condition 2): for $\Delta[n] \xrightarrow{x} X$ to belong to the set $H(X/U)_n$

the diagram

$$\begin{tikzcd}
& \Delta[n] \arrow[r, "x"] & X \arrow[r, "f"] & U \arrow[dr, "r"] & \\
\Delta[n] \arrow[ur, "\Delta[j_0]"] \arrow[dr, "\Delta[j_1]"'] & & /// & & X \\
& \Delta[n] \arrow[urrr, "x"'] & & &
\end{tikzcd}$$

commutes.

This means that if $x : x_0 \sim x_1$, then $x_1 = rf(x_0)$.

It is easy to check that the subsets

$$H(X/U)_n \subseteq H(X)_n \quad (n \in \mathbb{N})$$

so defined determine a subcomplex $H(X/U) \subseteq H(X)$.

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Special case: $H(U/U)$ for

$$\begin{tikzcd} U \arrow[d, "1_U = f"] \\ U \arrow[u, bend left, "1_U = r"] \end{tikzcd}$$

The set of trivial (identity) homotopies of U.

Under the general assumption of

$$\begin{tikzcd} X \arrow[d, "f"] \\ U \arrow[u, bend left, "r"] \end{tikzcd} \quad frf = f$$

the restriction of $H(f) : H(X) \to H(U)$

to $H(X/U)$, by the very definition of $H(X/U)$, factors through

$$H(U/U) \xrightarrow[\text{inclusion}]{f} H(U)$$

thus we have $H(f) : H(X/U) \longrightarrow H(U/U)$

by abuse of notation-

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$\boxed{\text{Proposition 1}}$ Suppose we have $X \underset{r}{\overset{f}{\rightleftarrows}} U$, $frf = f$. Then

1) Assume that $f: X \to U$ is a Kan fibration.

Then $H(f): H(X/U) \longrightarrow H(U/U)$ is also a Kan fibration.

2) Modify the definition $H(X/U)$ as indicated on pages 6.1, 6.2 and obtain $H^q(X/U)$. Assume that $f: X \to U$ is a qKan fibration ($f \in (\vec{I}_1)^\perp$; see earlier notes for $\vec{I}_1$). Then

$$H^q(X/U) \longrightarrow H(U/U) \text{ is a qKan fibration.}$$

—.—

proof: later

$\boxed{5}$ Lemmas, notation

$X \overset{i}{\rightarrowtail} Y$, i is a monomorphism (in $Set^{\Delta^{op}}$)

$X \overset{i}{\twoheadrightarrow\!\!\!\rightarrow} Y$: i is a strict adonyne map

($i \in Tc\, P_0(I_1)$)

$X \overset{i}{\twoheadrightarrow\!\!\!\rightarrow} Y$: i adonyne ; $i \in Re\, Tc\, P_0\, (I_1) = {}^{\perp}(I_1{}^{\perp})$

$X \underset{q}{\overset{i}{\twoheadrightarrow\!\!\!\rightarrow}} Y$: $i \in Tc\, P_0\, (\vec{I}_1)$; i is strict q-anodyne (stronger than strict anodyne)

$X \underset{q}{\overset{i}{\rightarrowtail}} Y$: $i \in Re Tc P_0\, (\vec{I}_1) = {}^{\perp}(I_1{}^{\perp})$

i q-adonyne

$X \overset{f}{\twoheadrightarrow} Y$: f is a Kan fibration : $f \in I_1{}^{\perp}$

$X \underset{q}{\overset{f}{\twoheadrightarrow}} Y$: f is a quasi-Kan fibration : $f \in (\vec{I}_1)^{\perp}$

$X \overset{f}{\twoheadrightarrow\!\!\!\twoheadrightarrow} Y$: f is a trivial fibration : $f \in I_0{}^{\perp}$

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$\boxed{\text{Lemma 2}}$

1) $$\begin{tikzcd} X \arrow[d, two heads, "f"'] \\ U \arrow[u, tail, bend right, "i"'] \end{tikzcd}$$, $fi = 1$, f ~~triv~~ fib $\Rightarrow$

i is strict q+ anodyne,

$$U \overset{i}{\underset{q}{\rightarrowtail}} X$$

hence also

$$U \overset{i}{\underset{\sim}{\rightarrowtail}} X$$

<u>Proof: later.</u>

(historically, the earliest piece in this work; done years ago)

$\boxed{\text{Lemma 3}}$

$$\begin{tikzcd} & X \arrow[dr, two heads, "f"] & & & X \arrow[dr, two heads, "f"] & \\ V \arrow[ur, tail, "g"] \arrow[dr, tail, "h"'] & \equiv & U & \Rightarrow & & U \\ & Y \arrow[ur, tail, two heads, "p"'] & & & & \end{tikzcd}$$

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In words:

$fg = ph$, g adonyne, h adonyne, p triv fib,

f Kan $\Rightarrow$ f triv fib

q - version:

$fg = ph$, $\overset{g}{\underset{q}{\rightarrowtail}}$; $\overset{h}{\underset{q}{\rightarrowtail}}$ (g, h: q-adonyne

p triv, fib. $\to$ q $\overset{f}{\underset{q}{\twoheadrightarrow}}$ (f q-Kan).

$\Rightarrow$ f triv fib

Proof: later - but: remarks: the proof uses Proposition 1 (in the special case $rf = 1_U$). Otherwise, it directly uses the (weak) universal properties of adonyne maps:

$$g, h \perp I_1^{\perp} \quad (\text{resp. } g, h \perp \vec{I}_1^{\perp})$$

without the analysis $g, h \in \mathrm{Retc}\,\mathrm{Po}(I_1)$

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$\boxed{\Pi 7}$

Recall : $X \xrightarrow{f} Y \in W$ (weak equivalence)

$$\underset{\text{def}}{\Longleftrightarrow} \exists : \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "g"'] & \arrow[d, phantom, "\equiv"] & Y \\ & Z \arrow[ur, two heads, "p"'] & \end{tikzcd} \quad f = pg,\ p \text{ triv fib},\ g \text{ strict adonyne}$$

Lemma ~~2~~ 4 The composite $U \overset{p}{\twoheadrightarrow} Y \overset{j}{\rightarrowtail} V$, is a weak equivalence

$U \xrightarrow{jp} Y$ with p triv fib, j strict adonyne

$$\Rightarrow U \xrightarrow{jp} Y \in W$$

Proof: Factor jp as $U \overset{m}{\rightarrowtail} S \overset{t}{\twoheadrightarrow} Y$, with m strict adonyne, t Kan.

We have

$$\begin{tikzcd} U \arrow[r, two heads, "p"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\equiv"] & Y \arrow[d, tail, "j"] \\ S \arrow[r, two heads, "t"'] & V \end{tikzcd} \quad . \quad tm = jp .$$

We show that t is triv. fib.; this will suffice.

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$\boxed{18}$

Consider the commutative square: $\emptyset$ is the empty complex (initial object in $Set^{\Delta^{op}}$)

$$\begin{tikzcd}
\emptyset \arrow[r, "!"] \arrow[d, tail, "!_Y"'] & U \arrow[d, two heads, "p"] \\
Y \arrow[r, "1_Y"'] \arrow[ur, dashed, "q"] & Y
\end{tikzcd}$$

There $\underbrace{!_Y}_{\text{monomorphism}}$ $\perp$ the p ($\in I_0^1$) ; there is $q : Y \to U$

such that $\boxed{pq = 1_Y.}$ By Lemma 1 : q is strict adonyne

$$Y \overset{q}{\rightarrowtail\!\!\!\!\!\twoheadrightarrow} U$$

As a composite of two strict adonyne's, mq is strict adonyne

claim:

$$\begin{tikzcd}
 & Y \arrow[dl, "mq"'] \arrow[d, "j"] \\
S \arrow[r, two heads, "t"'] \arrow[ur, phantom, "\equiv"] & V
\end{tikzcd}$$

commutes : $\boxed{tmq \overset{?}{=} j}$

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precompose this with p

$$\begin{tikzcd}
\textcircled{tm}qp \arrow[r, phantom, "\overset{?}{=}\checkmark"] \arrow[d, phantom, "\parallel"] & \dot{j}p \arrow[d, phantom, "\parallel\checkmark"] \\
\textcircled{jp}qp \arrow[r, phantom, "="] & \dot{j}1_Y p
\end{tikzcd}$$

Thus $tmqp = \dot{j}p$

Now, precompose it with q

$$\begin{tikzcd}
tmqpq \arrow[r, phantom, "="] \arrow[d, phantom, "\parallel"] & \dot{j}pq \arrow[d, phantom, "\parallel"] \\
tmq \arrow[r, phantom, "\underset{\checkmark}{=}"] & \dot{j}
\end{tikzcd}$$

$(pq = 1_Y)$

claim: $\underset{\sim}{\text{done}}$

Now, use <u>Lemma 3</u> to:

$$\begin{tikzcd}
Y \arrow[r, tail, "1_Y"] \arrow[d, tail, "mq"'] \arrow[dr, phantom, "\cong"] & Y \arrow[d, tail, "\dot{j}"] \\
S \arrow[r, two heads, "t"'] & V
\end{tikzcd}$$

$(U = Y,\ p = 1_Y$

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$\boxed{\boxed{20}}$

It follows that t is triv. fib.: $S \xrightarrow{t} V_1$

□ Lemma 4

(using Lemmas 2 & 3)

$\boxed{\text{Lemma 5}}$ The composite of w.e's is w.e

$$\begin{tikzcd}
V \arrow[rr, bend left, "w_2 w_1"] \arrow[rd, "w_1"'] & & U \\
& Z \arrow[ru, "w_2"'] &
\end{tikzcd}$$

$w_1, w_2 \in W \overset{?}{\Rightarrow} w_2 w_1 \in W$

By the definition of "w.e." : $w_1 = pl$, $w_2 = qi$; i, l : s.a. ; p, q : triv. fib

$$\begin{tikzcd}
& & & X \arrow[rrd, two heads, "f"] & & \\
V \arrow[rrru, tail, "g"] \arrow[rd, tail, "l"'] & & & \equiv & & U \\
& X \arrow[rd, two heads, "p"'] & & & T \arrow[ru, two heads, "q"'] & \\
& & Z \arrow[rru, tail, "i"'] & & &
\end{tikzcd}$$

"small object arg": $w_2 w_1 = fg$

f Kan fib, g : s.a.

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Use Lemma 4 : find $X \overset{j}{\rightarrowtail} Y \overset{r}{\twoheadrightarrow} T$ such that

$$i p = r j$$

$$\begin{tikzcd}
 & & X \arrow[ddrr, two heads, "f"] & & \\
V \arrow[urr, tail, "g"] \arrow[dr, tail, "e"] & & \equiv & & \\
 & & Y \arrow[r, two heads, "r"] & T \arrow[r, two heads, "q"] & U \\
 & X \arrow[ur, tail, "j'"] \arrow[dr, two heads] & \cong & & \\
 & & Z \arrow[uur, tail, "i'"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & X \arrow[dr, two heads, "f"] & \\
V \arrow[ur, tail, "g"] \arrow[dr, tail, "jl"] & \cong & U \\
 & Y \arrow[ur, two heads, "qr"] &
\end{tikzcd}$$

By Lemma 3, f is triv. fib.

$$\begin{tikzcd} X \arrow[dr, two heads, "f"] & \\ & U \end{tikzcd}$$

$w_1 w_2 = fg \in W$

by def'n of W. $\square$ Lemma 5

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$\boxed{\text{Lemma}}$ If f is a Kan fibration and a weak equivalence, then it is a trivial fibration.

Proof. Suppose

$$\begin{tikzcd} X \arrow[rr, two heads, "f"] \arrow[dr, tail, "g"'] & \arrow[d, phantom, "\equiv"] & Y \\ & Z \arrow[ur, two heads, "h"'] & \end{tikzcd}$$

(f : K

(induction statement: P(z

Lifting 0-simp's: ~~By~~ (g-) induction on $z_0 \in Z_0$, we prove that $h z_0 = y_0$ can be lifted along f: there is $x_0 \in X_0$ s.t. $x_0 \xmapsto{f} y_0$. Since h is surjective on $Z_0 \to Y_0$, this will show that f is surjective on $X_0 \to Y_0$.

Base case: $z_0 = g x_0$. Then $x_0 \xmapsto{f} y_0$ $(f = hg)$; done

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$\boxed{23.1}$

<u>Induction step</u>: Suppose $z_1 \overset{z_{10}}{\underset{\text{either/or}}{\longleftrightarrow}} z_0$, $z_1 < z_0$.

Let $y_1 = hz_1$. <u>Induction hypothesis</u>: $P(z_1)$ holds.

There is $x_1 \in X_0$, $x_1 \xmapsto{f} y_1 \underset{\text{def}}{=} hz_1$. Let $y_{10} \underset{\text{def}}{=} h(z_{10})$

In the context of the Kan fibration f, we have

$$\begin{tikzcd} x_1 \arrow[d, "f"'] & \\ y_1 \arrow[r, leftrightarrow, "y_{10}"'] & y_0 \end{tikzcd}$$

Therefore, there is $x_1 \overset{x_{10}}{\longleftrightarrow} x_0$ s.t.

$$\begin{tikzcd} x_1 \arrow[rr, leftrightarrow, "x_{10}"] & {} \arrow[d, "f"] & x_0 \\ y_1 \arrow[rr, leftrightarrow] & {} & y_0 \end{tikzcd}$$

in particular, $x_0 \mapsto y_0$ as desired.

$\boxed{\text{Lifting 0-simp's along } f\text{: done}}$

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Lifting 1-simp's: Suppose: $x_0, x_1 \in X_0$, $y_0 = fx_0$, $y_1 = fx_1$

$y_0 \xrightarrow{y_{01}} y_1$; want: there is $x_0 \xrightarrow{x_{01}} x_1$

such that $x_{01} \overset{f}{\longmapsto} y_{01}$.

Let: $z_0 = gx_0$, $z_1 = gx_1$; then $z_0 \overset{h}{\longmapsto} y_0$, $z_1 \overset{h}{\longmapsto} y_1$.

P: Since h is tf, there is z_{01}, $z_0 \xrightarrow{z_{01}} z_1$ s.t. $z_{01} \overset{h}{\longmapsto} y_{01}$.

We perform an induction on z_{01}. To be very precise, here is the induction statement on 1-simp's on $\mathbf{Z}$:

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$P(z_0 \xrightarrow{z_{01}} z_1) \equiv \underbrace{\text{for all}}_{\sim\sim}\ x_0, x_1 \in X_0$ such that

$x_i \xmapsto{f} h z_i \ (i=0,1)$, there is $x_0 \xrightarrow{x_{01}} x_1$ such that

$x_{01} \xmapsto{f} h z_{01}$.

(note: no direct connection between z_i and x_i!)

Proof by g-induction of $P(z_{01})$.

→ Suppose z_{01} is of rank 0. This means that, for some $\bar{x}_0, \bar{x}_1 \in X_0$,

$z_i = g\bar{x}_i \ (i=0,1)$ and for some $\bar{x}_{01} : \bar{x}_0 \to \bar{x}_1$ in X such that

$g(\bar{x}_{01}) = z_{01}$. Let $x_0, x_1 \in X_0$ arbitrary such that $x_i \xmapsto{f} h z_i \ (i=0,1)$

Main point: since $g x_i = z_i = g\bar{x}_i$, it follows that $x_i = \bar{x}_i$ since g is injective

Thus, $\bar{x}_{01} : x_0 \to x_1$; $f(\bar{x}_{01}) = hg(\bar{x}_{01}) = h(z_{01}) = y_{01}$.

Thus, $x_{01} \underset{\text{def}}{=} \bar{x}_{01}$ works.

→ Recall that ([illegible], [illegible], that [illegible]) :

$Z_1 = g(X_1) \dot{\cup}\, Z_1^0 \,\dot{\cup}\, Z_1^1 \,\dot{\cup}\, D_1$, and $Z_0^1 \underset{z \longmapsto z^+}{\cong} Z_1^0$; also $Z_1^0 \cap X_0 = \emptyset$

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$\operatorname{rank}(z_{01}) = 0 \iff z_{01} \in g_1 X_1$

$\rightsquigarrow$ Next, assume: $z_{01} \in Z_1^0$. Then $z_{01} \neq z^+$, $dz \overset{z^+}{\longleftrightarrow} cz$, and at least one of dz, cz is in Z_1^0; but $z_{01} : z_0 \to z_1$

i.e. $z_{01} : g x_0 \to g x_1$, with $g = \text{ind.}$, $z_{01} : x_0 \to x_1$

$$x_0 \xrightarrow{z_{01}} x_1$$

$$dz \xrightarrow{z_{01}} cz$$ with at least one of $dz, cz \notin X_0$: contradiction!

In other words, $z_{01} \in Z_1^0$ is impossible!

$\rightsquigarrow$ Assume $z_{01} \in D_1$; i.e., $z_{01} = 1_{z_0} : z_0 \to z_0$, $1_{z_0} : g x_0 \to g x_0$

But again, this is a contradiction, since D_1 consists of those degeneracies whose source is <u>not</u> $g(X)$ – done!

$\boxed{\text{one of three case, of which Case 1 is}}$

$\rightsquigarrow$ Finally, assume $z_{01} \in Z_1^1$. Then we have $z_2 \in Z_0$, $z_{02} : z_0 \to z_2$, $z_{21} : z_2 \to z_1$, $z_{021} \in Z_2$ such that

z_{02}, z_{21} are earlier than z_{01} $(z_{02}, z_{21} \prec z_{01})$

(they are predecessors of z_{01})

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and:
$$\begin{tikzcd}
 & z_2 \arrow[dr, "z_{21}"] & \\
z_0 \arrow[ur, "z_{02}"] \arrow[rr, "z_{01}"'] & \textcircled{z_{021}} & z_1
\end{tikzcd}$$
(i.e., z_{021} is a 2-simp, with 1-faces as shown)

Apply h, to obtain

$$\begin{tikzcd}
 & y_2 \arrow[dr, "y_{21}"] & \\
y_0 \arrow[ur, "y_{02}"] \arrow[rr, "y_{01}"'] & y_{021} & y_1
\end{tikzcd}$$
in Y

$y_2 = hz_2$

$y_{02} = hz_{02}$

$y_{21} = hy_{21}$

$y_{021} = hz_{021}$

The induction hypothesis itself tells us that $P(z_{02})$, $P(z_{01})$ are true. We already know that 0-simp's can be lifted along f; let $x_2 \in X_0$ be such that $x_2 \xmapsto{f} y_2$. Note that z_2 and x_2 may have nothing to do with each other. However, reading what $P(z_{02})$ says:

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$P(z_0 \xrightarrow{z_{02}} z_2)$: since $x_0 \overset{f}{\longmapsto} hz_0 = y_0$ and $x_2 \overset{f}{\longmapsto} hz_2 = y_2$

there is $x_0 \xrightarrow{x_{02}} x_2$ s.t. $x_{02} \longmapsto hz_{02} = y_{02}$

Similarly, for z_{21} in place of z_{02}. Thus, for $f : X \longrightarrow Y$

we have:

$$\begin{tikzcd}
 & x_2 \arrow[dr, "x_{21}"] & & & & y_2 \arrow[dr, "y_{21}"] & \\
x_0 \arrow[ur, "x_{02}"] & & x_1 \arrow[rr, mapsto, "f"] & & y_0 \arrow[ur, "y_{02}"] \arrow[rr, bend right, "y_{01}"'] & y_{021} & y_1
\end{tikzcd}$$

By f being a Kan fibration, there is x_{01} & x_{021} s.t.

$$\begin{tikzcd}
 & x_2 \arrow[dr, "x_{21}"] & & & \\
x_0 \arrow[ur, "x_{02}"] \arrow[rr, dashed, bend right, "x_{01}"'] & \circled{x_{021}} & x_1 \arrow[rr, mapsto] & & y_{021};
\end{tikzcd}$$

in particular, $f(x_{01}) = y_{01} = h(z_{01})$,

which shows that $P(z_{01})$ is true.

We have proved that $P(z_{01})$ is true for all $z_{01} \in Z_1$.

Now, to show the 1-lifting for $f: X \to Y$, , let: $x_i \xmapsto{f} y_i$ $(i=0,1)$, $y_{01}: y_0 \to y_1$ (in Y);

we want $\boxed{\substack{x_{01}\\ ?}}$ s.t. $x_{01} \xmapsto[?]{f} y_{01}$. Use that h

is t. triv. fib. to; $z_0 = g x_0$, $z_1 = g x_1$ – and thus $h z_i =$

$= h g x_i = f x_i = y_i$) —; get: $z_{01} : z_0 \to z_1$ in Z

s.t. $h z_{01} = y_{01}$. $P(z_{01})$ is true; this implies

that there is x_{01} s.t. $x_{01} \xmapsto{f} y_{01}$ as desired

($P(z_{01})$ was used in the special case when $g x_i = z_i$;

however, for the induction, we needed a stronger form of $P(z_{01})$!)