Sept 20/016

T1

2. $\omega$-graph $X$:

$\{X_n\}_{n \in \mathbb{N}}$

$$X_n \underset{c_n}{\overset{d_n}{\rightrightarrows}} X_{n-1}$$

$(n > 0)$ such that, for all $n > 1$:

$$d_n \, d_{n+1} = d_n \, c_{n+1}$$

$$c_n \, d_{n+1} = c_n \, c_{n+1}$$

i.e.:

commutative diagrams:

$$\begin{tikzcd} X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "c_{n+1}"'] \arrow[dr, phantom, "\equiv"] & X_n \arrow[d, "d_n"] \\ X_n \arrow[r, "d_n"'] & X_{n-1} \end{tikzcd} \qquad \begin{tikzcd} X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "c_{n+1}"'] \arrow[dr, phantom, "\equiv"] & X_n \arrow[d, "c_n"] \\ X_n \arrow[r, "c_n"'] & X_{n-1} \end{tikzcd}$$

more simply:

$dd = dc$

$cd = cc$

$$\begin{tikzcd} X_{n+1} \arrow[r, "d"] \arrow[d, "c"'] \arrow[dr, phantom, "\equiv"] & X_n \arrow[d, "d"] \\ X_n \arrow[r, "d"'] & X_{n-1} \end{tikzcd}$$ , etc.

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T2

<u>Notation:</u>

(**) $$a \xrightarrow{f} b$$

$f \in X_n$, some $n > 0$,

$a = d(f)$ $\quad (= d_n(f))$

$b = c(f)$;

$$\begin{tikzcd} a \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow x"] & b \end{tikzcd}$$

$f = d(x)$

$g = c(x)$

and, of course, $a = d(f) = d(g)$

$= (dd)(x) = (dc)x$

$b = \cdots$ (etc.)

$$\begin{array}{ccc} & \xrightarrow{\qquad f \qquad} & \\ a & x\Big\downarrow \xrightarrow{\ \varphi\ } \Big\downarrow y & b \\ & \xrightarrow[\qquad g \qquad]{} & \end{array}$$

$$x = d\varphi, \quad f = \overbrace{d^2\varphi = d^{(1)}\varphi}^{\text{notation}}$$

$$b = c^3\varphi = c^{(0)}\varphi$$

etc.

Arranged in a single picture:

$$X = \left[\; X_0 \underset{c}{\overset{d}{\longleftleftarrows}} X_1 \underset{c}{\overset{d}{\leftleftarrows}} \cdots \underset{c}{\overset{d}{\leftleftarrows}} X_{n-1} \underset{c}{\overset{d}{\leftleftarrows}} X_n \underset{c}{\overset{d}{\leftleftarrows}} X_{n+1} \leftleftarrows \cdots \right.$$

$$dd = dc$$
$$cd = cc$$

whenever meaningful

# Morphism of ($\omega$-) graphs

$$\frac{X \xrightarrow{F} Y}{\langle X_n \xrightarrow{F_n} Y_n \rangle_{n \in \mathbb{N}}}$$

such that: for all $n > 0$:

$$\begin{tikzcd} X_n \arrow[r, "F_n"] \arrow[d, "d"'] \arrow[dr, phantom, "\equiv"] & Y_n \arrow[d, "d"] \\ X_{n-1} \arrow[r, "F_{n-1}"'] & Y_{n-1} \end{tikzcd} \qquad d_n F_n = F_{n-1} d_n$$

$$\begin{tikzcd} X_n \arrow[r, "F_n"] \arrow[d, "d"'] \arrow[dr, phantom, "\equiv"] & Y_n \arrow[d, "c"] \\ X_{n-1} \arrow[r, "F_{n-1}"'] & Y_{n-1} \end{tikzcd} \qquad c_n F_n = F_{n-1} d_n$$

$X$ : $\omega$-graph

Notation:

For $a \in X_n$, $k \leq n$ :

$$d^{(k)}(a) = a^k \underset{\text{def}}{=} \underbrace{d}_{1} \cdots \underbrace{d}_{n-k}(a) ;$$

alternative notation

$$\boxed{\dim(a^k) = k}$$

$$c^{(k)}(a) = a_k \underset{\text{def}}{=} \cdots$$

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For convenience, let $X_{-1} = \{*\}$; and

for $a \in X_0$, $da \underset{\text{def}}{=} ca \underset{\text{def}}{=} *$.

Thus, any two 0-cells are <u>parallel</u>.

— · —

<u>Notation:</u> let $m, n \in \mathbb{N}$;

$a \in X_m$ $(\Leftrightarrow \dim(a) = m)$

$b \in X_n$

and $k \; (= k[a, b]) \underset{\text{def}}{=} \min(m, n) - 1$.

$a$ and $b$ are <u>composable</u> if T6

$m, n \geq 1$ & $a_k = b^k$.

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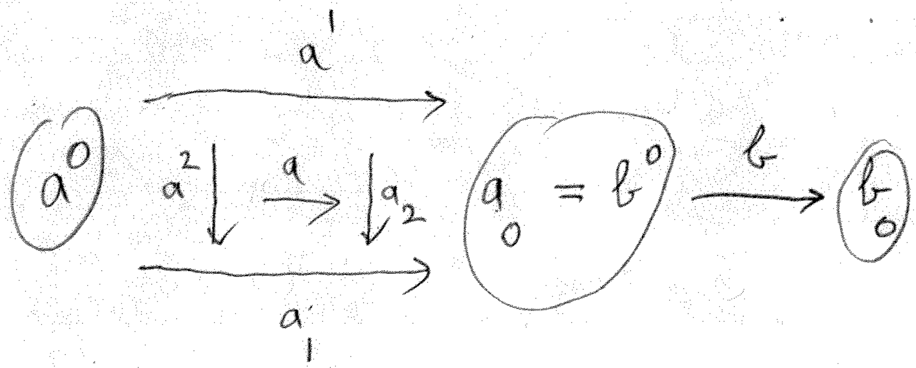
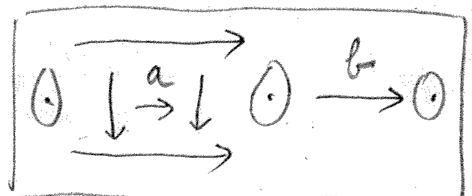
<u>Example</u>: $m = 3$, $n = 1$; $\therefore k = 0$





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For $m, n \geq 1$,

$$X_{m,n} \underset{\text{def}}{=} \left\{ (a, b) \in X_m \times X_n : a \ \& \ b \text{ are composable} \right\}$$

Definition of (strict) $\omega$-category $X$ has:

(i) Underlying: $\omega$-graph $X$;

(ii) Operations: (other than the $d$, $c$ ...):

for each pair $m, n$ positive ($\geq 1$):

$$\boxed{\bullet_{m,n} : X_{m,n} \longrightarrow X_{\max(m,n)}}$$

for each $r \geq 0$:

$$\boxed{1_{(-)} : X_r \longrightarrow X_{r+1}}$$

NB: Notation: diagrammatic:

for $(a, b) \in X_{m,n}$, $a \cdot b = ab = a \cdot_{m,n} b$
$\in X_{\max(m,n)}$

example: $m = n = 1$:

$$\cdot \xrightarrow{a} \cdot \xrightarrow{b} \cdot \quad \longmapsto \quad \cdot \xrightarrow{ab} \cdot$$

(T8)

'usually' : $a \cdot b \quad = \quad b \circ a$

(iii) Laws :

(1) domain / codomain laws :

$$\underline{d}(1_a) = \underline{c}(1_a) = a$$

$$\underline{d}(ab) = \begin{cases} (\underline{d}a)b & \text{if } m > n \\ a(\underline{d}b) & \text{if } m < n \\ \underline{d}a & \text{if } m = n \end{cases}$$

$$\underline{c}(ab) = \begin{cases} (\underline{c}a)b & \text{if } m > n \\ a(\underline{c}b) & \text{if } m < n \\ \underline{c}b & \text{if } m = n \end{cases}$$

(2) unit laws :

$$1_a b = \begin{cases} b & \text{if } 0 \leq m < n \\ 1_{ab} & \text{if } m \geq n \geq 1 \end{cases}$$

$$a\,1_b = \begin{cases} a & \text{if } m > n \geq 0 \\ 1_{ab} & \text{if } 1 \leq m \leq n \end{cases}$$

3) ASSOCIATIVE LAWS:

Let: $a \in X_m$, $b \in X_n$, $e \in X_p$

$(m, n, p \geq 1)$.

Assume: either $m = n \leq p$,

or $m \geq n = p$

or $m = p \leq n$

and both $ab$ and $be$ are well-defined

Then, inductively, both $a(be)$ and $(ab)e$ are well-defined and parallel. (*)

Require: $\boxed{a(be) = (ab)e}$

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(*) $u$ & $v$ are parallel $\iff$ $\dim(u) = d(v)$

& $d(u) = d(v)$, $c(u) = c(v)$

$$\bullet \underset{v}{\overset{u}{\rightrightarrows}} \bullet$$

T10

# 4) DISTRIBUTIVE LAWS:

Let (again) : $a \in X_m$, $b \in X_n$, $e \in X_p$

$(m, n, p \geq 1)$

## 4.1) Assume : $m < n$, $m < p$

and

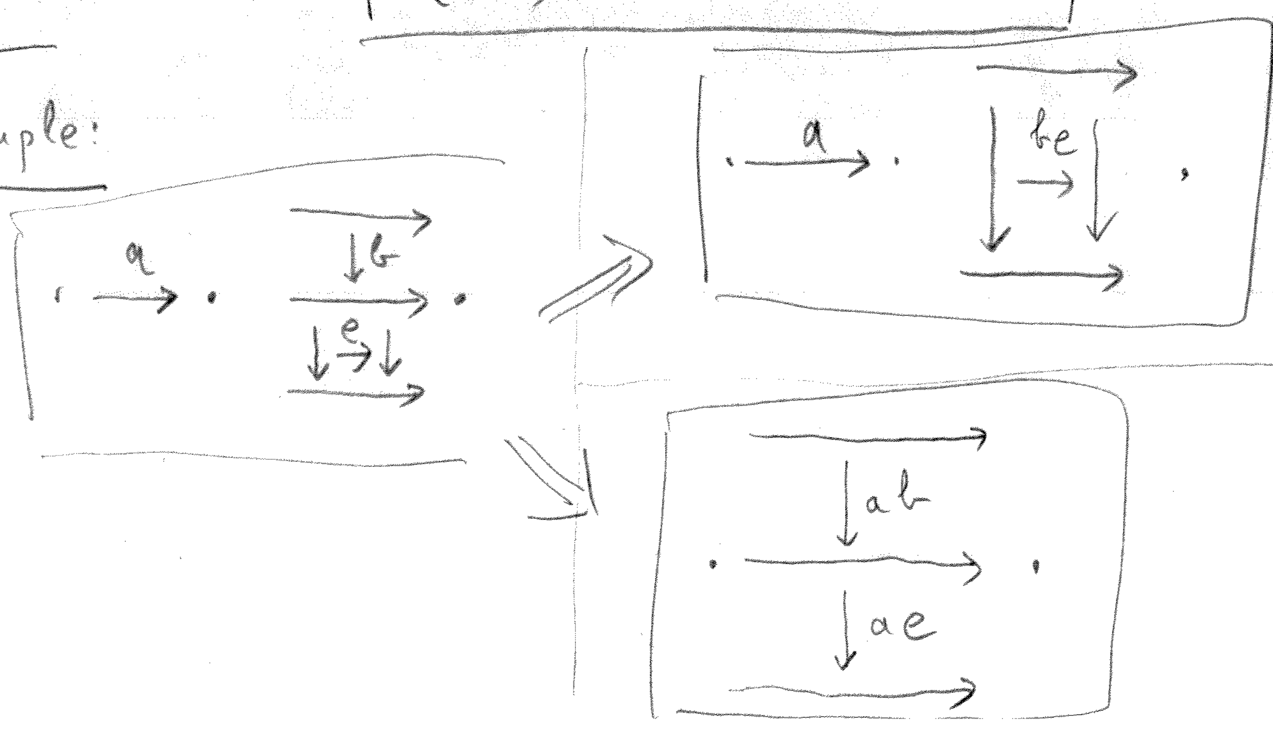
$ab$, $ae$, $be$ well-defined

Then, inductively, both $a(be)$ and $(ab)(ae)$ are well-defined and parallel.

Require:

$$\boxed{a(be) = (ab)(ae)}$$

Example:



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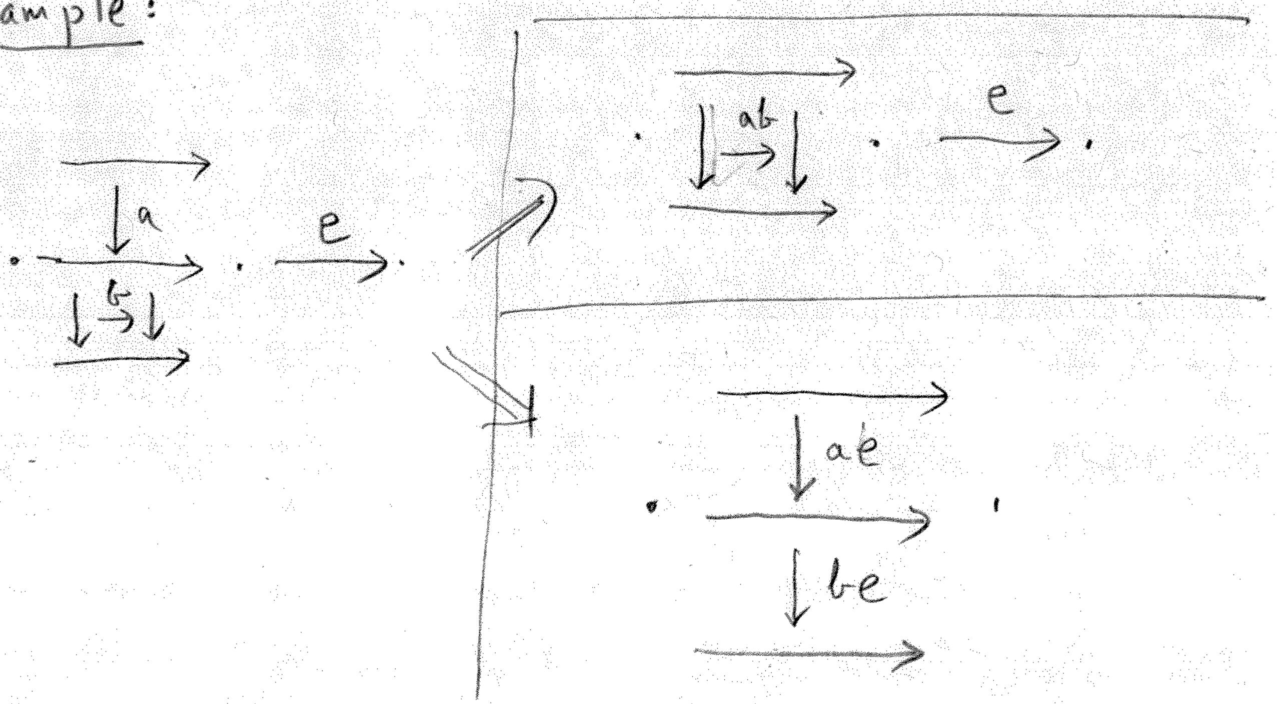
## (4.2) Dual to 4.1):

Assume: $p < m, \quad p < n$

(...) $\boxed{(ab)e = (ae)(be)}$

Example:




## 5) COMMUTATIVE LAWS:

Let: $a \in X_m, \quad b \in X_n \quad (m, n \geq 2)$

$$k \underset{\mathrm{def}}{=} k[a,b] = \min(m,n) - 1 \quad (\geq 1)$$

Assume:

$$c^{(k-1)} a = d^{(k-1)} b$$

Require:

$$(a(d^{(k)} b))((c^{(k)} a) b) = ((d^{(k)} a) b)(a(c^{(k)} b))$$

Example 1: $m = n = 2$:

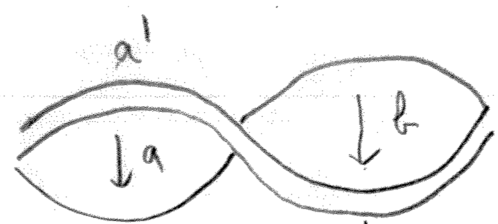
$$\begin{tikzcd} \bullet \arrow[r, bend left, "a'"] \arrow[r, bend right, "a_1"'] \arrow[r, phantom, "\Downarrow a"] & \bullet \arrow[r, bend left, "b'"] \arrow[r, bend right, "b_1"'] \arrow[r, phantom, "\Downarrow b"] & \bullet \end{tikzcd}$$

gives rise to:

$\underline{a}|\overline{b} \underset{\text{def}}{=} (a b')(a_1 b)$ :


and

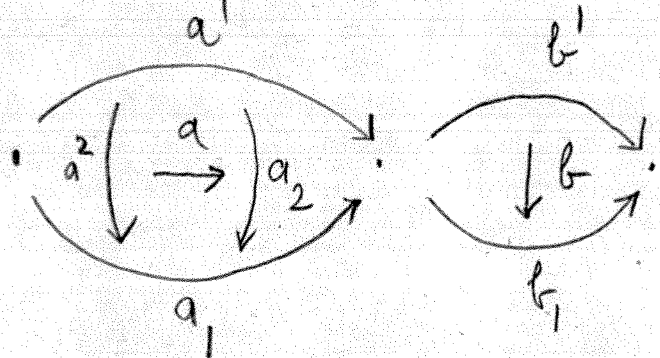
$\overline{a}|\underline{b} \underset{\text{def}}{=} (a' b)(a b_1)$ :




both: $\begin{tikzcd} \bullet \arrow[r, bend left, "a'b'"] \arrow[r, bend right, "a_1 b_1"'] \arrow[r, phantom, "\Downarrow \underline{a}|\overline{b} \qquad \Downarrow \overline{a}|\underline{b}"] & \bullet \end{tikzcd}$ : parallel

## Example 2:

$m = 3, \quad n = 2$




$k = 1$

now, the axiom is:

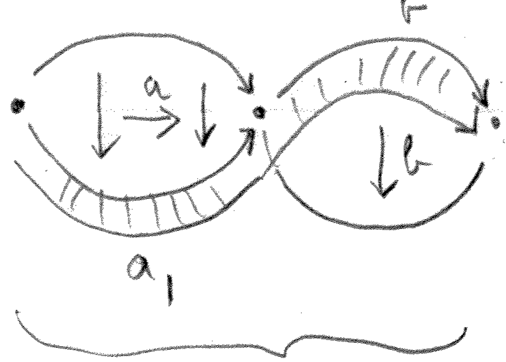
$$(a b')(a_1 b) = (a' b)(a b_1)$$

dimensions: 3 2 2 3

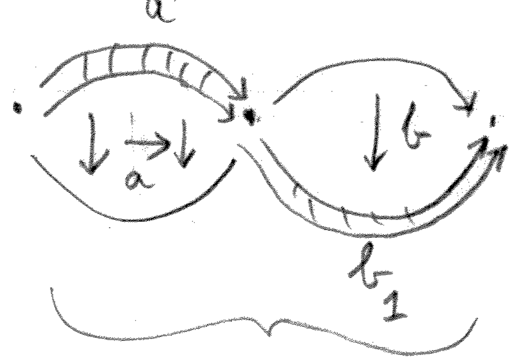
Denote LHS by $\underline{a}|\overline{b}$

RHS by $\overline{a}|\underline{b}$

Pictures:




$\underline{a}|\overline{b}$




$\overline{a}|\underline{b}$

$\underline{a}\rfloor b$ and $\overline{a}\rceil\underline{b}$ <u>well-defined</u>,

because

$$\left.\begin{array}{l} c^1(a\,b') = a_1 b^1 \\ d^1(a, b) = a_1 b^1 \end{array}\right\} \text{equal}$$

and similarly for $\overline{a}\rceil\underline{b}$

Moreover, $\underline{a}\rfloor b$ and $\overline{a}\rceil\underline{b}$ are <u>parallel</u>, since

$$\left.\begin{array}{l} d(\underline{a}\rfloor b) = \underline{a^2}\rfloor b \\ d(\overline{a}\rceil\underline{b}) = \overline{a^2}\rceil\underline{b} \end{array}\right\} \text{equal (!)}$$

by the 2-2 case of the commutative law

Similarly for $c$ ...

End of definition of (strict) $\omega$-category