Notes for MM's
talks on Oct 18 & 25, 2016

(in two parts)

the $\boxed{\text{totality}}$ of all natural numbers: $\mathbb{N}$ a set	several sets: $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, $\mathbb{Q}$, $\mathbb{R}$, ...	'all' sets and connecting them : $\in$
The $\boxed{\text{totality}}$ of all sets $\mathbb{V}$ in one way – for us: Set, the category of sets	Several categories Gp, Top, ...	'all' categories ⓪ and connecting them functors ① natural transf's ②
The $\boxed{\text{totality}}$ of all categories: CAT: the 2-category of categories or AnaCAT: the bicategory of cats & ana's	several 2-categories: Reg, Exact, FinLim, Coh, Acc, Pretop, BoolePretop, Heyting, ...	'all' 2-cats / bicats ⓪ and connecting 2-functors / pseudofunctors ① 2-nat transfs / psnt's ② modifications ③
The $\boxed{\text{totality}}$ of all 2-dim cats: possibilities: 2-CAT GRAY BICAT a 3-dim cat (degree's of 'weak'ness)	several 3-dim cats: e.g.: the Gray-category of finitely locally presentable 2-cat	... 'all' Gray-cats, ...

Motto:

"Today I looked at my work and was so struck by its lucidity that I wrote another article".

(in the "revolutionary magazine" De Stijl; cca 1919)

(PIET MONDRIAN, the painter)

Quoted in: "Mondrian from figuration to abstraction"

Exhibition, The Hague, 1988.

$\boxed{\mathbb{C} \text{ cartesian closed:}}$

binary products, terminal object

and

$$A, S \longmapsto [A,S] \quad \text{(objects);}$$

plus $\langle \varphi_C \rangle_C$ system of bijections

$$\varphi_C : \hom(C, [A,S]) \xrightarrow{\cong} \hom(C \times A, S);$$

illustrated:

$$\frac{C \xrightarrow{f} [A,S]}{C \times A \xrightarrow[g]{} S} \;\updownarrow \varphi_C$$

satisfying <u>naturality</u> in C: for all $D \xrightarrow{r} C$:

$$\frac{D \xrightarrow{r} C \xrightarrow{f} [A,S]}{D \times A \xrightarrow[r \times A]{} C \times A \xrightarrow{g} S} \quad \varphi_C \; \varphi_C \; \varphi_D$$

Examples:

1. A Heyting algebra ($\approx$ intuitionistic propositional logic) is a lattice $(\leq, \bot, \top, \wedge, \vee)$, which is Cartesian closed as a category.

2. Set is Cartesian closed: $[A, S] =$ set all functions $A \to S$

the
$$\frac{C \xrightarrow{f} [A,S]}{C \times A \underset{g}{\longrightarrow} S}$$
adjunction

is given by:
$$\boxed{g(c,a) = f(c)(a)} \quad (c \in C,\ a \in A)$$

3. CAT, the category of 'large' categories ($\mathrm{Set} \in \mathrm{CAT}$) is Cartesian closed;

$[A, S]$ is the category whose arrows are the functors $A \xrightarrow{F} S$, and morphisms $F \xrightarrow{h} G$: $A \underset{G}{\overset{F}{\rightrightarrows}} S$ ($\downarrow h$) are the nat. transf.'s

$\boxed{\text{Joyal's theorem on intuitionism}}$

$\mathcal{T}$: small Heyting category

("what is that?")

Have: $\underbrace{\mathrm{Coh}(\mathcal{T})}_{\text{category of } \underline{\text{coherent}} \text{ functors } \mathcal{T} \to \mathbf{Set}} \overset{i}{\hookrightarrow} [\mathcal{T}, \mathbf{Set}]$

for $\varepsilon_{\mathcal{T}}$, see later: T_3 6.2, T_3 6.3

Have:

$$\begin{tikzcd}
\mathcal{T} \arrow[rr, "\varepsilon_{\mathcal{T}}"] \arrow[dr, "ev_{\mathcal{T}}"'] & & {[[\mathcal{T}, \mathbf{Set}], \mathbf{Set}]} \arrow[dl, "i^*"] \\
& {[\mathrm{Coh}(\mathcal{T}), \mathbf{Set}]} &
\end{tikzcd}$$

$e_{\mathcal{T}} \overset{\text{def}}{=} i^* \circ ev_{\mathcal{T}}$

$\boxed{\underline{\text{Theorem}} \text{ (André Joyal)}}$ (see: MM – G.E. Reyes: LNM 611, 1978(?))

$$e_{\mathcal{T}} : \mathcal{T} \longrightarrow [\mathrm{Coh}(\mathcal{T}), \mathbf{Set}]$$

is a conservative (faithful) $\underline{\text{Heyting}}$-morphism.

Saul Kripke's completeness theorem for intuitionism

T : intuitionistic (Heyting) first-order theory

Kripke structure ((potential) model) for T:

M : poset $(P, \leq)$

$p \in P$: M_p

ordinary (Set-)
structure in the lang. of T

$p \leq q$: $h_{p,q} : M_p \to M_q$

"Tarski-style" we <u>define</u>

$$p \Vdash \varphi(\vec{a}) \qquad (\vec{a} \in M_p)$$

(e.g., $p \Vdash \forall x\, \varphi(x, \vec{a})$

$$\iff$$

for all $q \geq p$, and all $b \in M_q$

$q \Vdash \varphi(b, h_{p,q}(\vec{a}))$)

(cont'd on T_3 4.1)

$$M \models \sigma \overset{}{\underset{\text{def}}{\iff}} \bot \Vdash_M \sigma$$

<u>Kripke completeness</u>:

$$T \underset{\substack{\text{intuitionistic} \\ \text{formal logic} \\ \text{deducibility}}}{\vdash} \sigma$$

$$\iff$$

for all Kripke structures M for T

$$M \models T \implies M \models \sigma$$

Adjunction in general:

Suppose:
categories & functors:

$$\mathbb{A} \underset{G}{\overset{F}{\underset{\perp}{\leftrightarrows}}} \mathbb{X}$$

We say: F is left adjoint to G, $F \dashv G$,
G is right adjoint to F

if there is a system $\langle \varphi_{A,X} \rangle_{A \in \mathbb{A},\, X \in \mathbb{X}}$

of bijections

$$\frac{X \longrightarrow GA}{FX \longrightarrow A} \; \varphi_{A,X} \updownarrow$$

satisfying, for all $X' \xrightarrow{x} X$ in $\mathbb{X}$, $A \xrightarrow{a} A'$ in $\mathbb{A}$:

$$\frac{X' \xrightarrow{x} X \longrightarrow GA \xrightarrow{Ga} GA'}{FX' \xrightarrow[Fx]{} FX \longrightarrow A \xrightarrow[a]{} A'}$$

$\varphi_{X,A}$

$\varphi_{X',A'}$

If so, we have nat. transf.'s

$$\varepsilon : FG \longrightarrow 1_{\mathbb{A}}$$

$$\eta : 1_{\mathbb{X}} \longrightarrow GF$$

where

$$\frac{G(A) \xrightarrow{1_{GA}} G(A)}{FG(A) \xrightarrow[\varepsilon_A]{} A} \; \varphi_{A, G(A)}$$

and

$$\frac{X \xrightarrow{\eta X} GF(X)}{F(X) \xrightarrow[1_{F(X)}]{} F(X)} \; \varphi_{F(X), X}$$

We have:

$$\begin{tikzcd} FG \arrow[r, "\varepsilon"] & \mathbb{A} \arrow[loop above, "1_{\mathbb{A}}"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\perp"] & & \mathbb{X} \arrow[ll, bend right, "F"'] \arrow[loop above, "1_{\mathbb{X}}"] \arrow[r, "\eta"] & GF \end{tikzcd}$$

satisfying the "triangle equalities":

$$\begin{tikzcd}[column sep=small] & FGF \arrow[dr, "\varepsilon F"] & \\ F \arrow[ur, "F\eta"] & = & F \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & GFG \arrow[dr, "G\varepsilon"] & \\ G \arrow[ur, "\eta G"] & = & G \end{tikzcd}$$

$$(\varepsilon F) \circ (F\eta) = 1_F \qquad (G\varepsilon) \circ (\eta G) = 1_G$$

i.e., for all $X \in \mathbb{X}$

$$\begin{tikzcd} & FGF(X) \arrow[dr, "\varepsilon_{F(X)}"] & \\ F(X) \arrow[ur, "F(\eta_X)"] \arrow[rr, "1_{F(X)}"'] & \equiv & F(X) \end{tikzcd}$$

and similarly for the other

The contravariant exponential "self"-adjunction

$\underline{\mathbb{C}}$: Cartesian closed ; $S \in \mathbb{C}$.

Have:

$$\begin{tikzcd} \underline{\mathbb{C}}^{op} \arrow[rr, bend right, "{[-, S]}"'] \arrow[rr, phantom, "\perp"] & & \underline{\mathbb{C}} \arrow[ll, bend right, "{[-, S]}"'] \end{tikzcd}$$

⊙ $[-, S] : \underline{\mathbb{C}}^{op} \to \underline{\mathbb{C}}$ (or: $\underline{\mathbb{C}} \longrightarrow \underline{\mathbb{C}}^{op}$)

is a (contravariant) functor:

$$\begin{array}{c} A \\ f \uparrow \\ B \end{array} \text{ in } \underline{\mathbb{C}} : \quad \mapsto \quad \begin{array}{c} [A, S] \\ \downarrow [f, S] \\ [B, S] \end{array} = f^* \text{ for short ; ?}$$

definition of f^*

$$\dfrac{[A, S] \xrightarrow{f^* = g^\wedge} [B, S]}{\dfrac{[A, S] \times B \xrightarrow{g} S}{[A, S] \times B \xrightarrow{[A, S] \times f} [A, S] \times A \xrightarrow{e_A} S}} \quad \varphi_{[A,S]} \quad \updownarrow$$

$$g \stackrel{def}{=} e_A \circ ([A, S] \times f)$$

where (next page)

$$\frac{[A,S] \xrightarrow{1} [A,S]}{[A,S]\times A \xrightarrow[e_A]{} S}\ \varphi_A$$

⊙ the adjunction:

$$\begin{tikzcd} \underline{\mathbb{C}}^{op} \arrow[r, bend right, "{[-,S]}"'] \arrow[r, phantom, "\perp"] & \underline{\mathbb{C}} \arrow[l, bend right, "{[-,S]}"'] \end{tikzcd}$$

$(A \in \underline{\mathbb{C}})$ $(X \in \underline{\mathbb{C}})$

$([X,S] \to A \text{ in } \underline{\mathbb{C}}^{op})$
$$\boxed{\frac{A \longrightarrow [X,S]}{X \longrightarrow [A,S]}}\ ? \quad \text{in } \underline{\mathbb{C}}$$

as follows:

$$\cfrac{\cfrac{A \longrightarrow [X,S]}{A\times X \longrightarrow S}}{\cfrac{X\times A \longrightarrow S}{X \longrightarrow [A,S]}}\ \gamma$$

NB: $\gamma_{A\times X} : A\times X \xrightarrow{\cong} X\times A$ 'switching';
natural in A and X.

The counit:

$$\varepsilon : FG \longrightarrow 1_{\mathbb{C}^{op}}$$

$$\varepsilon_A : A \longrightarrow [[A, S], S] \qquad (A \in \mathbb{C})$$

(contravariance!)

and unit:

$$\eta_X : X \longrightarrow [[X, S], S] \qquad (X \in \mathbb{C})$$

are essentially the same.

<u>The derivation of</u> $\underline{\varepsilon}$:

$$\begin{array}{c|c}
A \xrightarrow{\varepsilon_A} [[A, S], S] & \uparrow \\
\hline
A \times [A, S] \longrightarrow S & \uparrow \\
\hline
[A, S] \times A \xrightarrow[e_A]{} S & \\
\hline
[A, S] \xrightarrow[1_{[A,S]}]{} [A, S] & \uparrow
\end{array}$$

Example:

$\underline{\mathbb{C}} = CAT$, $S \in CAT$

For $T \in CAT$

$$T \xrightarrow{\varepsilon_T} [[T, S], S]$$

(evaluation; already used on T_3 3) is:

$$X \longmapsto \varepsilon_T(X) = [F \longmapsto F(X)]$$

$$\boxed{\varepsilon_T(X)(F) = F(X)}$$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd}[column sep=huge] {[T, S]} \arrow[r, bend left, "\varepsilon_T(X)"] \arrow[r, bend right, "\varepsilon_T(Y)"'] \arrow[r, phantom, "\Downarrow \varepsilon_T(f)"] & S \end{tikzcd}$$

nat. transf.

$$F \longmapsto \begin{tikzcd} F(X) \arrow[d, "F(f)"] \\ F(Y) \end{tikzcd}$$

$\mathbb{C}$ is $\boxed{\text{regular}}$ if

(1) it has finite limits
(terminal object and pullbacks: enough)

(2) whenever

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ K \arrow[ur, "k_1"] \arrow[dr, "k_2"'] \arrow[rr, phantom, ">"] & & B \\ & A \arrow[ur, "f"'] & \end{tikzcd}$$

is a pullback

((k_1, k_2) is a kernel pair), the coequalizer

$$K \underset{k_2}{\overset{k_1}{\rightrightarrows}} A \xrightarrow{g} C$$

exists. Automatic: (k_1, k_2) is a kernel-pair of g.

g is a regular epi (def'n)

(3) regular epi's are stable under pullback

$$\begin{tikzcd} A \arrow[r, "g"] & C \\ A' \arrow[u] \arrow[r, "g'"'] \arrow[ur, phantom, "\lrcorner"] & C' \arrow[u] \end{tikzcd}$$

pullback & g reg. epi $\Rightarrow$ g' reg. epi.

Functor $\mathbb{C} \longrightarrow \mathbb{S}$ is regular, if it preserves finite limits and reg. epis.

$\mathrm{Reg}[\mathbb{C}, \mathbb{S}] \underset{\text{full}}{\hookrightarrow} [\mathbb{C}, \mathbb{S}]$

$\mathbb{C}$ is

(Barr) exact if it is regular, and

every equivalence relation

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A$$

is a kernel pair.

For every regular $\mathbb{C}$, there is an exact completion, $\mathbb{C} \xrightarrow{i} ex(\mathbb{C})$

such that for every exact $\mathbb{S}$ (such as Set)

$$\mathrm{Reg}[ex(\mathbb{C}), \mathbb{S}] \xrightarrow[\simeq]{i^*} \mathrm{Reg}[\mathbb{C}, \mathbb{S}]$$

is an equivalence of categories.