Gödel completeness for coherent categories

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1. Coherent categories

Definition (i) Category $\mathbb{C}$ is coherent (called "logical" in [MR](= "First Order Categorical Logic", SLNM 611 (1977)) if$_{def}$ 1.1) — 1.5) below hold:

1.1) $\mathbb{C}$ has finite limits; equivalently, a terminal object, and a pullback of any fork

$$\begin{array}{c} A \xrightarrow{f} C \\ \uparrow g \\ B \end{array} \quad \dashrightarrow \quad \begin{tikzcd} A \arrow[r, "f"] & C \\ A \underset{C}{\times} B = P \arrow[u, dashed, "\pi_0"] \arrow[r, dashed, "\pi_1"] & B \arrow[u, "g"] \end{tikzcd}$$

1.2) Every morphism $A \xrightarrow{f} B$ can be factored as $f = hg$, where h is a mono(morphism), g is an extremal epi(morphism) (e.e., for short), where

$$A \xrightarrow{g} C$$

is e.e. if$_{def}$ every time $g = ml$ & m is mono:

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "l"'] & \arrow[d, phantom, "\circleddash"] & C \\ & D \arrow[ur, "m"'] & \end{tikzcd} \quad \Longrightarrow \quad m \text{ is an isomorphism.}$$

1.3) E.e.'s are stable under pullback:

$$\begin{tikzcd} A \arrow[r, two heads, "g"] & C \\ A \times_C C' = A' \arrow[u] \arrow[r, "g'"'] \arrow[ur, phantom, "\text{pb}"] & C' \arrow[u] \end{tikzcd}$$

& g is e.e. $\Rightarrow$ g' is e.e.

(1.4) Each subobject $(\wedge, \top)$-semilattice $\mathrm{Sub}(X)$ is a lattice: has a bottom element $\perp_X \in \mathrm{Sub}(X)$ (bottom = least), and for $\underline{A}, \underline{B} \in \mathrm{Sub}(X)$, we have a join (least upper bound) $\underline{A} \vee \underline{B} \in \mathrm{Sub}(X)$.

[Note: The elements of $\mathrm{Sub}(X)$ are equivalence classes $[A, a]$ of monomorphisms $(A, a: A \rightarrowtail X)$ into X; the equivalence relation is:

$$(A, a) \sim (B, b) \iff$$

there is a (necessarily unique) isomorphism $f: A \xrightarrow{\cong} B$ such that

$$\begin{tikzcd} A \arrow[dr, tail, "a"] \arrow[d, "f", "\cong"'] & \\ B \arrow[r, tail, "b"'] \arrow[ur, phantom, "\circledcirc"] & X \end{tikzcd}$$

$\mathrm{Sub}(X)$ is a poset via the relation $[A, a] \leq [B, b]$ defined as:

$[A, a] \leq [B, b] \iff$ there is (unique) $f: A \to B$ such that

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, tail, "a"] \arrow[d, phantom, shift left=4ex, "\equiv"] & \\ B \arrow[r, tail, "b"'] & X \end{tikzcd} .$$

$[X, 1_X]$ is the top element, $\top_X$, of $\mathrm{Sub}(X)$. The meet (greatest lower bound) of $\underline{A} = [A, a]$ and $\underline{B} = [B, a]$ is given by the pullback:

$$\begin{tikzcd} & A \arrow[dr, tail, "a"] & \\ C \arrow[ur, tail] \arrow[dr, tail] \arrow[rr, tail, dashed, "c"] & & X \\ & B \arrow[ur, tail, "b"'] & \end{tikzcd} ;$$

$\underline{A} \wedge \underline{B} = \underline{C} \underset{\text{def}}{=} [C, c]$. The requirement 1.4) has

$\underline{A} \vee \underline{B} = \underline{D} = [D, d]$ so that

$$\begin{tikzcd} A \arrow[dr, tail] \arrow[drr, tail, bend left, "a"] & & \\ & D \arrow[r, tail, "d"] & X \\ B \arrow[ur, tail] \arrow[urr, tail, bend right, "b"'] & & \end{tikzcd}$$

and every time

$$\begin{tikzcd} A \arrow[dr, tail] \arrow[drr, tail, bend left, "a'"] \arrow[drrr, tail, bend left=40, "a"] & & & \\ & D & E \arrow[r, tail, "e"] & X \\ B \arrow[ur, tail] \arrow[urr, tail, bend right, "b'"'] \arrow[urrr, tail, bend right=40, "b"'] & & & \end{tikzcd}$$

, we have $D \rightarrowtail E$

to make

$$\begin{tikzcd}
A \arrow[dr, tail] \arrow[drr, bend left, "a'"] & & \\
& D \arrow[r, tail, dashed] & E \\
B \arrow[ur, tail] \arrow[urr, bend right, "b'"'] & &
\end{tikzcd}$$

commute.].

1.5 Finite sups are pullback-stable:

for all $X \xrightarrow{f} Y$, we have $f^* : \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$

defined by:
$$\begin{tikzcd}
B \arrow[d, tail, "b"] \\
Y
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\underset{B}{A\times Y} = A \arrow[r] \arrow[d, tail, "a"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, tail, "b"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd};$$

$$[B, b] \xmapsto{f^*} [A, a]$$

Require: f^* preserves (sups):

$$f^*(\bot_Y) = \bot_X$$

$$f^*(\underline{B} \vee_Y \underline{B}') = f^*\underline{B} \vee_X f^*\underline{B}'$$

end of definition
of coherent category

<u>Examples</u>: 1) Set, the category of sets, is coherent.

Set is <u>Boolean</u>, meaning (for an arbitrary coherent $\mathbb{C}$) that each subobject lattice $\mathrm{Sub}_{\mathbb{C}}(X)$ is a Boolean algebra: for every $\underline{A} \in \mathrm{Sub}(X)$, there is (necessarily unique) $\underline{A}' \in \mathrm{Sub}(X)$ such that

$$\underline{A} \wedge \underline{A}' = \bot_X \quad \text{and} \quad \underline{A} \vee \underline{A}' = \top_X .$$

2) Whenever $\mathbb{C}$ is coherent, and $\mathbb{G}$ is any small category, then the functor category $\mathbb{C}^{\mathbb{G}}$ (consisting of the functors $\mathbb{G} \to \mathbb{C}$ as objects, and the natural transformations as arrows) is coherent. In fact, the coherent structure of $\mathbb{C}^{\mathbb{G}}$ is <u>component-wise</u>: <u>for instance</u>, a diagram

$$\begin{tikzcd} A \arrow[r, "f"] & C \\ P \arrow[u, "\pi_0"] \arrow[r, "\pi_1"'] & B \arrow[u, "g"'] \end{tikzcd} \qquad (*)$$

in $\mathbb{C}^{\mathbb{G}}$ is a pullback diagram if and only if for each object $G \in \mathbb{G}$, the diagram

$$\begin{tikzcd} A(G) \arrow[r, "f_G"] & C(G) \\ P(G) \arrow[u, "(\pi_0)_G"] \arrow[r, "(\pi_1)_G"'] & B(G) \arrow[u, "g_G"'] \end{tikzcd} \qquad (**)$$

is a pullback diagram. Also, with any given fork

$$\begin{tikzcd} A \arrow[r, "f"] & C \\ & B \arrow[u, "g"'] \end{tikzcd}$$

in $\mathbb{C}^{\mathbb{G}}$, when one chooses a pullback diagram (**) in $\mathbb{C}$, for each G, arbitrarily, then there is a unique way to complete the definition of the functor $P: \mathbb{G} \longrightarrow \mathbb{C}$ on arrows so as to make (*) commutative ((and, I should've said, π_0, π_1 <u>natural transformations</u>) — <u>and</u>, this diagram (*) is then a pullback diagram. What we say for this is that the evaluation functors

$$\mathbb{C}^{\mathbb{G}} \xrightarrow[e_G]{} \mathbb{C}$$

$$\begin{array}{ccc} A & \longmapsto & A(G) \\ f \downarrow & \longmapsto & \downarrow f_G \\ B & \longmapsto & B(G) \end{array}$$

jointly <u>create</u> pullbacks in $\mathbb{C}^{\mathbb{G}}$.

Similarly: an arrow $A \xrightarrow{f} B$ in $\mathbb{C}^{\mathbb{G}}$ is e.e. iff for all $G \in \mathbb{G}$, $e_G(f)$ is e.e. in $\mathbb{C}$.

Consider the example $Set^{\mathbb{G}}$, where $\mathbb{G} = \mathbb{2}$, the category $\boxed{0 \longrightarrow 1}$ (the order 2; identities suppressed in the notation). $Set^{\boxed{0\to 1}}$ is coherent (and a lot more!) but it is not Boolean!

2. Coherent functors

For coherent categories $\mathbb{C}$ and $\mathbb{D}$, a functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ is <u>coherent</u> (called "logical" in G11) if F preserves finite limits, extremal epi's and finite sups of subobjects:

2.1.1 ⊙ $\quad F(\underbrace{1_{\mathbb{C}}}_{\text{terminal object in } \mathbb{C}}) \cong \underbrace{1_{\mathbb{D}}}_{\text{terminal object in } \mathbb{D}}$.

2.1.2 ⊙
$$\begin{tikzcd} A \arrow[r, "f"] & C \\ P \arrow[u, "\pi_0"] \arrow[r, "\pi_1"'] \arrow[ur, phantom, "\lrcorner"] & B \arrow[u, "g"'] \end{tikzcd} \quad \underline{\text{pb in } \mathbb{C}} \quad \Longrightarrow \quad \begin{tikzcd} FA \arrow[r, "Ff"] & FC \\ FP \arrow[u, "F\pi_0"] \arrow[r, "F\pi_1"'] & FB \arrow[u, "Fg"'] \end{tikzcd}$$

is a <u>pullback in $\mathbb{D}$</u>.

(2.1.3) ⊙ $A \overset{f}{\twoheadrightarrow} B$ e.e. in $\mathbb{C}$

$$\Rightarrow \quad FA \xrightarrow{Ff} FB \text{ e.e. in } \mathbb{D};$$

(2.1.4) ⊙ $\underline{A}, \underline{B} \in \mathrm{Sub}_{\mathbb{C}}(X)$;

$$F(\underline{A} \vee_X \underline{B}) = F\underline{A} \vee_{FX} F\underline{B};$$

(2.1.5) ⊙ $$F(\bot_X) = \bot_{FX}.$$

Small coherent categories and coherent functors form a category called $\underline{\mathrm{Coh}}$: a subcategory of $\underline{\mathrm{Cat}}$, the category of small categories (with arrows the functors): the identity functor $\mathbb{C} \xrightarrow[\mathrm{Id}_{\mathbb{C}}]{} \mathbb{C}$ is coherent ($\mathbb{C}$: coherent cat), and the composite of coherent functors

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[rr, bend right, "GF"'] & \mathbb{D} \arrow[r, "G"] & \mathbb{E} \end{tikzcd}$$

is again coherent.

It is good to consider $\underline{\mathrm{COH}}$, the category of possibly large coherent categories, in which $\underline{\mathrm{Set}}$, for instance, is an object. (More precisely: we have Grothendieck universes $\underline{U}_0$, $\underline{U}_1$ and $\underline{U}_2$; $\underline{U}_0$ is an <u>element</u> of $\underline{U}_1$;

Set is the category of $\underline{U}_0$-small sets (sets that are elements of $\underline{U}_0$), then Set is an element of $\underline{U}_1$; Coh is the category of $\underline{U}_0$-small coherent cat's; Coh $\in \underline{U}_1$; COH is the category of $\underline{U}_1$-small coherent categories; Set is an object of COH; Coh is a full subcategory of COH; etc...).

Coh and COH are in fact 2-categories – but we are not looking at their 2-cat structures now.

Returning to examples:

Fin Set:

the category of finite sets is a coherent category (in fact, a pretopos) – and although not small but it is equivalent to a small category

$\mathbb{N}$

whose objects are the natural numbers, conceived of as von-Neumann ordinals (n is $= \{0, \dots, n-1\}$)

An equivalence functor $\mathbb{C} \to \mathbb{D}$ (full and faithful, essentially surjective on objects) is always coherent ($\mathbb{C}, \mathbb{D}$: coherent categories).

Returning to the example

$$\mathbb{C}^{\mathbb{G}}$$

($\mathbb{C}$: coherent) ; the evaluations

$$e_G : \mathbb{C}^{\mathbb{G}} \longrightarrow \mathbb{C} \qquad (G \in \mathbb{G})$$

are coherent functors.

When $\mathbb{C}$ is a small coherent category, a coherent functor $M : \mathbb{C} \longrightarrow \mathbf{Set}$ is called a <u>model</u> of $\mathbb{C}$.

<u>Theorem</u> (Gödel completeness theorem)

Let $\mathbb{C}$ be a small coherent category. Then, for any arrow $f : A \longrightarrow B$ in $\mathbb{C}$, the following holds:

f is an isomorphism (it has a two-sided inverse)

if and only if

for all models $M : \mathbb{C} \to \mathbf{Set}$

$M(f)$ is an isomorphism

($M(f) : M(A) \longrightarrow M(B)$ is a bijection)

<u>Remarks 1</u>:- Consider the definition of "coherent category" without the 'stability' conditions 1.3) and 1.5); call the resulting notion "pre-coherent category". Note that the notion of 'coherent functor' makes sense, without change, for pre-coherent categories. Now, assume that $\mathbb{C}$ is pre-coherent, and the assertion of the completeness theorem holds for it. <u>Show</u> that $\mathbb{C}$ is necessarily coherent: the stability conditions 1.3) and 1.5) hold for $\mathbb{C}$.

2. It is clear that 'completeness' asserts that there are "many models" of a small coherent $\mathbb{C}$: in fact, given any non-isomorphism arrow f, <u>there must be</u> at least one model M such that $M(f)$ is a non-isomorphism. Accordingly, the proof of the completeness theorem is expected to consist mainly in constructing 'many' models of $\mathbb{C}$. We will carry out the proof – following the important early work of Hugo Volger – in the form of a series of constructions of objects and of arrows, starting with the single object $\mathbb{C}$, in the category <u>COH</u>

Let me note that the proof of the same theorem that can be found in [MR] '6.11' is entirely different: it is a "logician's proof".

3. The 'local' constructions 1 :

when is the global-sections functor coherent?

3.1 Definitions. Let $\mathbb{C}$ be a coherent category.

3.1.1 We say $\mathbb{C}$ is complete if $\top = \top_{1_{\mathbb{C}}}$, the top element of $\mathrm{Sub}(1_{\mathbb{C}})$ ($1_{\mathbb{C}}$: 'the' terminal object of $\mathbb{C}$) is finitely irreducible: whenever I is a finite set (possibly empty), $U_i \in \mathrm{Sub}(1_{\mathbb{C}})$ $(i \in I)$ and

$$\underbrace{\bigvee_{i\in I}}_{\text{supremum}} U_i = \top$$

there is $i \in I$ such that $U_i = \top$.

NB: 1) For $I = \emptyset$, this says: $\top \neq \bigvee\emptyset = \bot =$ bottom elt.

empty set of subobjects

2) It is enough to require condition for $I = \emptyset$ and $I = 2$.

3.1.2 $\mathbb{C}$ is saturated if$_{\text{def}}$ for all $X \in \mathrm{Ob}(\mathbb{C})$ such that $!_X : X \twoheadrightarrow 1_{\mathbb{C}}$ is e.e. (extremal epi), there is an arrow of the form $x : 1_{\mathbb{C}} \longrightarrow X$ (a global element of X) (of course, usually there are several global elements)

NB: 1) If $x : 1_{\mathbb{C}} \longrightarrow X$, then $!_X$ is e.e.

Let us see a (more) general statement first:

if $A \xrightarrow{f} B$ has a right inverse:

$$\begin{tikzcd} A \arrow[r, "f"'] & B \arrow[l, bend right, "g"'] \end{tikzcd} \qquad fg = 1_B$$

then f is e.e. (Such f is a "split epi".)

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"'] & & B \arrow[ll, bend right, "g"'] \\ & C \arrow[ur, "m"'] & \end{tikzcd}$$

Assume: m mono; $mh = f$,

to show m is an isomorphism.

We have: $m(hg) = (mh)g = fg = 1_B$.

Let $k = hg$; we have $\boxed{mk = 1_B}$; then

consider $m(km) = (mk)m = m1_C$.

$$\begin{tikzcd} C \arrow[r, shift left, "km"] \arrow[r, shift right, "1_C"'] & C \arrow[r, tail, "m"] & B \end{tikzcd}$$

Since m is mono, $\boxed{km = 1_C}$. Thus, $k = m^{-1}$.

When $x : 1_{\mathbb{C}} \longrightarrow X$, then $\begin{tikzcd} X \arrow[r, "!_X"'] & 1_{\mathbb{C}} \arrow[l, bend right, "x"'] \end{tikzcd}$: $!_X x = 1_{1_{\mathbb{C}}}$

(since the only arrow $1_{\mathbb{C}} \longrightarrow 1_{\mathbb{C}}$ is $1_{1_{\mathbb{C}}}$)

x is a right inverse to $!_X$ — $!_X$ is e.e.

(3.2) For $\underline{A} = [A, a] \in \mathrm{Sub}_{\mathbb{C}}(X)$, let us say that $x \in \underline{A}$ ("x is an element of $\underline{A}$") if $x : 1_{\mathbb{C}} \longrightarrow X$ is a global element of X, and there is a factorization

$$\begin{tikzcd}
1_{\mathbb{C}} \arrow[r, "x"] \arrow[dr, dashed] & X \arrow[d, phantom, "\circlearrowleft"] \\
 & A \arrow[u, tail, "a"']
\end{tikzcd}$$

Let us write $\Gamma(X)$ for the set of (global) elements of X. Then, <u>if $\mathbb{C}$ is complete</u>, $X \in \mathrm{Ob}(\mathbb{C})$, I is a finite set, $\underline{A}_i \in \mathrm{Sub}(X)$ $(i \in I)$,

for any $x \in \Gamma(X)$:

$$\boxed{x \in \bigvee_{i \in I} \underline{A}_i \iff \exists i \in I .\ x \in \underline{A}_i} \qquad (*)$$

The right-to-left direction $(\Leftarrow)$ is immediate, since $x \in \underline{A}$ and $\underline{A} \leq \underline{B}$ imply $x \in \underline{B}$. Let $[A, a] = \bigvee_{i \in I} \underline{A}_i = \underline{A}$, and assume $x \in \underline{A}$. We have:

$$\begin{tikzcd} \mathbb{1} \arrow[r, "x"] & X \\ x^*(A) \arrow[u, tail, "a'"] \arrow[r] \arrow[ur, "x'"] & A \arrow[u, tail, "a"'] \end{tikzcd}$$

when the square is constructed as a pullback. Use the pullback property to:

$$\begin{tikzcd} & \mathbb{1} \arrow[r, "x"] & X \\ & x^*(A) \arrow[u, tail, "a'"] \arrow[r] & A \arrow[u, tail, "a"] \\ \mathbb{1} \arrow[uur, bend left, "1_{\mathbb{1}}"] \arrow[ur, dashed, "g"] \arrow[urr, bend right, "x'"'] & & \end{tikzcd} \qquad x \cdot 1_{\mathbb{1}} = a x' \ (\checkmark)$$

resulting in an arrow $\mathbb{1} \xrightarrow{g} x^*(A)$ such that $a'g = 1_{\mathbb{1}}$

Since a' is a monomorphism, $ga' = 1_{\mathbb{1}}$ follows, and a' is an isomorphism.

This means that

$$x^*\left(\bigvee_{i \in I} \underline{A}_i\right) = \top_{\mathbb{1}} .$$

Condition 1.5) says: $x^*\left(\bigvee_{i\in I} \underline{A}_i\right) = \bigvee_{i \in I} x^*(\underline{A}_i) = \top_{\mathbb{1}}$

By completeness, there is $i \in I$ such that $x^*(\underline{A}_i) = \top$

We have:

$$\begin{tikzcd} \mathbb{1} \arrow[r, "x"] \arrow[dr, "x''"] & X \\ x^*(A_i) \arrow[u, tail, "a''", "\cong"'] \arrow[r, "y"'] & A_i \arrow[u, tail, "a_i"'] \end{tikzcd}$$

(square: pullback; a_i' : isomorphism)

$$x'' \underset{\text{def}}{=} y(a'')^{-1}$$

Then

$$\boxed{a_i x'' \overset{?}{=} x} \quad \checkmark$$

$$\| $$

because

$$a_i\, y(a'')^{-1} = (a_i y)(a'')^{-1} = (x\, a'')(a'')^{-1}$$

which shows that $\boxed{x \in A_i}$. (QED for (*), p 14)

We have the global-sections functor

$$\Gamma : \mathbb{C} \longrightarrow \mathbf{Set}$$

defined as the functor represented by the terminal object $\mathbb{1} = \mathbb{1}_{\mathbb{C}}$: $\Gamma = \mathbb{C}(\mathbb{1}, -)$.

This means:

$$\Gamma(X) = \text{the set of global elements of } X = \mathbb{C}(\mathbb{1}, X)$$

and for $\begin{array}{c} X \\ f\downarrow \\ Y \end{array}$: $\begin{array}{c} \mathbb{C}(\mathbb{1}, X) \\ \mathbb{C}(\mathbb{1}, f) \downarrow \\ \mathbb{C}(\mathbb{1}, Y) \end{array}$ $\begin{array}{c} x \\ \downarrow \\ fx \end{array}$ $\begin{array}{c} \mathbb{1} \xrightarrow{x} X \\ \downarrow \\ \mathbb{1} \xrightarrow{x} X \xrightarrow{f} X \end{array}$

for any object U of $\mathbb{C}$, the functor represented by U, $\mathbb{C}(U,-) : \mathbb{C} \longrightarrow \mathrm{Set}$ preserves all finite limits, and as a result, preserves monomorphisms. Thus, in particular, $\Gamma : \mathbb{C} \longrightarrow \mathrm{Set}$ induces, for every $X \in \mathrm{Ob}(\mathbb{C})$, a map

$$\Gamma_X : \mathrm{Sub}_{\mathbb{C}}(X) \longrightarrow \underbrace{\mathcal{P}(\Gamma(X))}^{\text{power-set of } \Gamma(X)}$$

$$\begin{array}{ccccc} A & & \Gamma(A) & & \\ a \big\downarrow & \longmapsto & \big\downarrow \Gamma(a) & \longmapsto & \text{the image of } \Gamma(a) \\ X & & \Gamma(X) & & (\text{a subset of } \Gamma(X)) \end{array}$$

(Note that $\mathcal{P}(S) \cong \mathrm{Sub}_{\mathrm{Set}}(S)$ for any set S where a monomorphism (injective mapping) $A \overset{m}{\rightarrowtail} S$ corresponds to the subset of S which is the image of m: $\{s \in S : \exists a \in A.\ m(a) = s\}$)

Just from the fact that Γ preserves finite limits, Γ_X preserves the meet-semi-lattice structure — and, the $\boxed{\text{box}}$ on p. 14 says that

$$\Gamma_X : \mathrm{Sub}_{\mathbb{C}}(X) \longrightarrow \mathcal{P}(\Gamma(X))$$

preserves the finite-sup structure, since, for $\underline{A} \in \mathrm{Sub}_{\mathbb{C}}(X)$, $\Gamma_X(\underline{A})$, a subset of $\Gamma(X)$, is precisely the set of <u>elements</u> of $\underline{A}$, as we defined those above.

<u>Conclusion:</u> When $\mathbb{C}$ is complete, then the global-sections functor $\Gamma : \mathbb{C} \to \mathrm{Set}$ induces lattice homomorphisms

$$\Gamma_X : \mathrm{Sub}_{\mathbb{C}}(X) \longrightarrow \mathcal{P}(\Gamma(X)).$$

(3.3) Suppose now that $\mathbb{C}$ is <u>saturated</u> (see: 3.1.2) and let $\Gamma : \mathbb{C} \to \mathrm{Set}$ be the global-sections functor. Then Γ preserves e.e. morphisms:

$A \xrightarrow{f} B$ e.e. in $\mathbb{C}$.

$$\Longrightarrow \quad \Gamma(f) : \Gamma(A) \longrightarrow \Gamma(B)$$

is e.e. (surjective) in Set

<u>Proof</u>.

$$\Gamma(f) : \Gamma(A) \longrightarrow \Gamma(B)$$

$$\begin{tikzcd} (\mathbb{1} \xrightarrow{a} A) \arrow[r, mapsto] & (\mathbb{1} \arrow[r, "a"] \arrow[rr, bend right, "fa"'] & A \arrow[r, "f"] & B) \end{tikzcd}$$

We have to show that for any $b : \mathbb{1} \to B$, there is

$a : 1 \longrightarrow A$ such that $b = fa$. Consider

$$\begin{tikzcd}
A \arrow[r, two heads, "f"] & B \\
C \arrow[u, "h"] \arrow[r, two heads] & 1 \arrow[u, "b"'] \arrow[ul, "a \overset{def}{=} hc"'] \arrow[l, bend left=40, "c", "g"']
\end{tikzcd}$$

First, we construct C, h and g as a pullback of f and b. By condition 1.3), g as a pullback of the e.e. arrow f, is itself e.e. This means that C has full support, and by saturation, there is $c : 1 \longrightarrow C$. We define 'a' as $a = hc$. We check

$$fa \overset{?}{=} b \;\checkmark$$

$$fa = f(hc) = (fh)c = (bg)c = b(gc) = b(1_1) = b$$

$gc : 1 \to 1$ must be the identity

$\square$

Summary of 3.2 and 3.3:

For $\mathbb{C}$ a coherent category that is complete and saturated, the global-sections functor

$$\Gamma = \mathbb{C}(1_{\mathbb{C}}, -) : \mathbb{C} \longrightarrow \mathrm{Set}$$

is a coherent functor.

(As Bill Boshuck pointed out, the converse is also true.)

4. The slice construction

Let $\mathbb{C}$ be a coherent category, and let us fix an object X of $\mathbb{C}$. We define $\mathbb{C}/X$ ($\mathbb{C}$ over X) as the category whose objects are pairs (A, a) where $a : A \to X$ (a any arrow, not just a monomorphism) and whose arrows

$$(A, a) \xrightarrow{f} (B, b)$$

are $A \xrightarrow{f} B$ such that the triangle

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "a"'] & \arrow[d, phantom, "\ominus"] & B \arrow[dl, "b"] \\ & X & \end{tikzcd}$$

commutes: $a = bf$.

Composition of arrows: just as in $\mathbb{C}$:

$$\begin{tikzcd}(A,a) \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & (B,b) \arrow[r, "g"] & (C,c)\end{tikzcd}$$

since we have:

$$\begin{tikzcd}A \arrow[r, "f"] \arrow[dr, "a"'] & B \arrow[d, "b"] \arrow[r, "g"] & C \arrow[dl, "c"] \\ & X & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd}A \arrow[r, "gf"] \arrow[d, "a"'] & C \arrow[dl, "c"] \\ X & \end{tikzcd}$$

The identity on (A,a) is $1_A : (A,a) \longrightarrow (A,a)$.

We have functors connecting $\mathbb{C}$ and $\mathbb{C}/X$:

$$\begin{tikzcd}\mathbb{C} \arrow[r, shift right, "\Phi"'] & \mathbb{C}/X \arrow[l, shift right, "\Psi"']\end{tikzcd}$$

$\Psi : \mathbb{C}/X \longrightarrow \mathbb{C}$ is the forgetful functor:

$$(A,a) \longmapsto A.$$

The more important functor is

$$\Phi : \mathbb{C} \longrightarrow \mathbb{C}/X$$

$$A \longmapsto (A \times X, \pi_1)$$

where we use 'the' product

$$\begin{tikzcd} & A \times X \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ A & & X\end{tikzcd}$$

On arrows, $\Phi : \mathbb{C} \longrightarrow \mathbb{C}/X$

acts as follows:

$$A \xrightarrow{f} B \longmapsto \begin{tikzcd} A\times X \arrow[rr, "f\times X"] \arrow[dr, "\pi_1"'] & \arrow[d, phantom, "\circledcirc \checkmark"] & B\times X \arrow[dl, "\pi_1"] \\ & X & \end{tikzcd}$$

$$\Phi(f) = f\times X$$

Reminder: $f\times X$ is determined, by the universal property of $B\times X$, by the commutativities in:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, phantom, "\circledcirc"] & B \\ A\times X \arrow[u, "\pi_0"] \arrow[r, "f\times X"'] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\circledcirc"] & B\times X \arrow[u, "\pi_0"'] \arrow[d, "\pi_1"] \\ X \arrow[r, "1_X"'] & X \end{tikzcd}$$

(easy exercise!)

Φ is a functor (check). In fact, Ψ is left adjoint to Φ — but I avoid(?) using this fact.

Proposition $\mathbb{C}/X$ is a coherent category, and

$$\Phi : \mathbb{C} \longrightarrow \mathbb{C}/X$$

is a coherent functor

Proof.

4. (1 ⓘ) The terminal object of $\mathbb{C}/X$ is $(X, 1_X)$:

indeed, for every (A, a), $(A,a) \xrightarrow{f} (X, 1_X)$:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "a"'] & \arrow[d, phantom, "="] & X \arrow[dl, "1_X"] \\ & X & \end{tikzcd}$$

is the unique $f = a$.

4. (2 ⓘ) Also, $\Phi : \mathbb{C} \longrightarrow \mathbb{C}/X$ sends

$\mathbb{1} = \mathbb{1}_{\mathbb{C}}$ to $(\mathbb{1} \times X, \pi_1 : \mathbb{1} \times X \to X)$ which is

isomorphic to $(X, 1_X)$:

$$\begin{tikzcd} \mathbb{1} \times X \arrow[rr, "\cong", "\pi_1"'] \arrow[dr, "\pi_1"'] & & X \arrow[dl, "1_X"] \\ & X & \end{tikzcd}$$

[The fact that Φ preserves (finite) limits (also) follows from Φ being a right adjoint.

4. (3 ⓘ) Pullbacks in $\mathbb{C}/X$ are calculated as in $\mathbb{C}$:

for the pullback:
$$\begin{tikzcd} (A,a) \arrow[r, "f"] & (C,c) \\ (P,p) \arrow[u, "\pi_0"] \arrow[r, "\pi_1"'] \arrow[ur, phantom, "\urcorner"] & (B,a) \arrow[u, "g"'] \end{tikzcd}$$

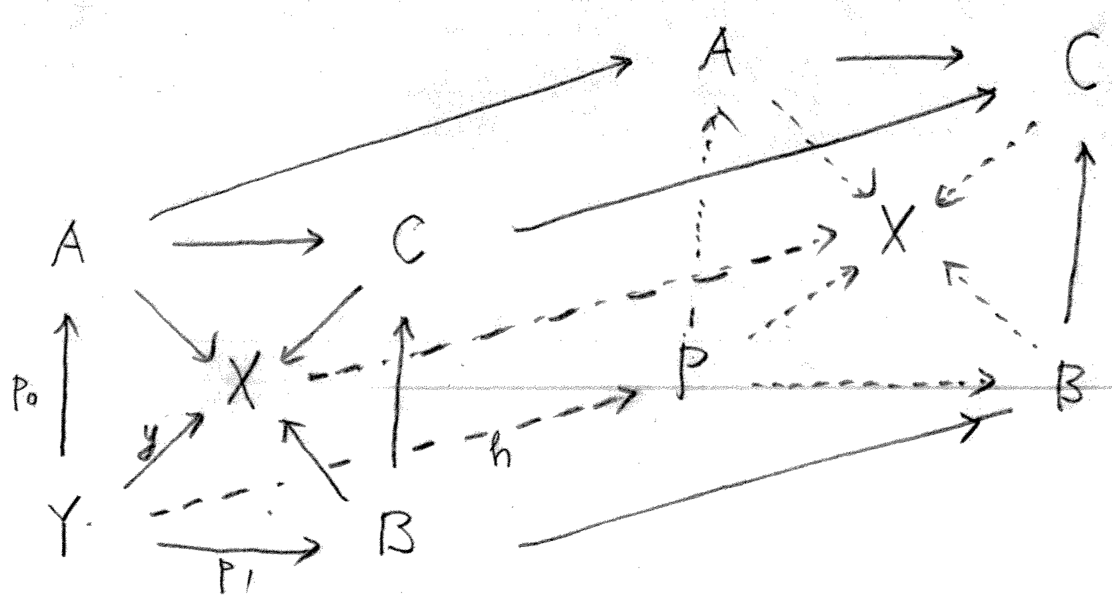
I can take

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "a"] \arrow[rr, phantom, shift right=3, "1"] & & C \arrow[dl, "c"'] \arrow[d, phantom, shift right=4, "2"] \\
& X & \\
P \arrow[uu, "\pi_0"] \arrow[rr, "\pi_1"'] \arrow[ur, dashed, "p"] & & B \arrow[uu, "g"'] \arrow[ul, "b"']
\end{tikzcd} \qquad (*)$$

where the square is a pullback in $\mathbb{C}$, and $p: P \longrightarrow X$ is defined as $p = a\pi_0 \underset{?}{=} b\pi_1$

where the last (necessary) equality follows from the commutativities 1 and 2, and the commutativity of the square.

I check the universal property: consider



The arrows $A \to A$, $B \to B$, $C \to C$, $X \to X$

(going from front to back) are identities. The names $a : A \to X$ and the like are taken from the top diagram, and not notated now. I have to explain those arrows that have Y as source (four of them).

Let (Y, y) be an object of $\mathbb{C}/X$, $p_0 : (Y, y) \to (A, a)$ $p_1 : (Y, y) \to (B, b)$ be such that the front square commutes:

$$f p_0 = g p_1 .$$

I have to show: there is unique $h : (Y, y) \to (P, p)$ such that $p_0 = \pi_0 h$, $p_1 = \pi_1 h$. Since P is a pullback in $\mathbb{C}$ (see (*)), there is a unique $h : Y \to P$ such that the two required equalities hold. I still need to show that $h : (Y, y) \to (P, p)$, i.e. :

$$y \overset{?}{=} p h$$

But: $y = a p_0$ since $p_0 : (Y, y) \to (A, a)$;

$p = a \pi_0$ since $\pi_0 : (P, p) \to (A, a)$;

and $p_0 = \pi_0 h$;

thus

$$y = a p_0 = a \pi_0 h = p h \quad \text{as desired.}$$

4.(40) Properties of the functor $\Psi : \mathbb{C}/X \to \mathbb{C}$:

4.(4.1) Ψ is faithful: obvious.

4.(4.2) Ψ is conservative: it reflects isomorphisms: let's write $\mathbb{D}$ for $\mathbb{C}/X$; this means:

$$f \in \mathrm{Arr}(\mathbb{D}),\ \Psi(f) \text{ is an iso (in } \mathbb{C}) \overset{?}{\Rightarrow} f \text{ is an iso.}$$

pf: let $f : (A, a) \to (B, b)$, and suppose that $\Psi(f) = f : A \to B$ is an isomorphism. There is $g : B \to A$ s.t. $gf = 1_A$, $fg = 1_B$. Need: $g : (B, b) \to (A, a)$ in $\mathbb{D}$; i.e.:

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "b"'] & \arrow[d, phantom, "\circledcirc ?"] & A \arrow[dl, "a"] \\ & X & \end{tikzcd} \qquad b \overset{?}{=} ag$$

But $a = bf$; thus $ag = (bf)g = b(fg) = b$. $\boxed{\text{Done}}$

<u>Rmk:</u> The proof shows that Ψ has a much stronger reflection property: it reflects 'right inverses':

g being a right inverse of f, $fg = 1_B$, implies that $g : (B, b) \to (A, a)$, and thus g is a right inverse of $f : (A, a) \to (B, b)$.

4. 4.3 ① Ψ preserves and reflects pullbacks:

given a diagram $\hat{\mathcal{D}}$

$$\hat{\mathcal{D}} :=: \begin{tikzcd} (A, a) \arrow[r, "f"] & (C, c) \\ (\hat{P}, \hat{p}) \arrow[u, "\hat{\pi}_0"] \arrow[r, "\hat{\pi}_1"'] & (B, b) \arrow[u, "g"'] \end{tikzcd}$$

in $\mathbb{D} = \mathbb{C}/X$, $\hat{\mathcal{D}}$ is a pullback if and only if $\Psi(\mathcal{D})$ (= obtained by applying Ψ to each of the eight entities in $\hat{\mathcal{D}}$) is a pullback in $\mathbb{C}$.

proof: 1) Suppose $\hat{\mathcal{D}}$ is a pullback (in $\mathbb{D}$). Then $\mathcal{D}$ is isomorphic to the pullback diagram $\mathcal{D}$ constructed:

$$\mathcal{D} :=: \begin{tikzcd} (A, a) \arrow[r, "f"] & (C, c) \\ (P, p) \arrow[u, "\pi_0"] \arrow[r, "\pi_1"'] & (B, b) \arrow[u, "g"'] \end{tikzcd}$$

above: there is $(P, p) \xrightarrow{i} (\hat{P}, \hat{p})$, an isomorphism such that

$$(1_A, 1_B, 1_C, i) : \mathscr{D} \xrightarrow{\cong} \hat{\mathscr{D}} \qquad (*)$$

[$\mathscr{D}$ and $\hat{\mathscr{D}}$ are, in fact, graph maps $\mathbb{G} \to \mathbb{D}$, for

$$\mathbb{G} :=: \begin{tikzcd} 0 \arrow[r, "5"] & 2 \\ 3 \arrow[u, "7"] \arrow[r, "8"'] & 1 \arrow[u, "6"'] \end{tikzcd}$$

graph-maps $\mathbb{G} \to \mathbb{D}$ form a category $[\mathbb{G}, \mathbb{D}]$, with arrows natural transformations; $(*)$ is a natural isomorphism in $[\mathbb{G}, \mathbb{D}]$]. Since any functor maps an iso to an iso, $\Psi(\mathscr{D}) \cong \Psi(\hat{\mathscr{D}})$. By construction, $\Psi(\mathscr{D})$ was a pullback; ~~an~~ diagram isomorphic to a pullback diagram is a pullback diagram; therefore, $\Psi(\hat{\mathscr{D}})$ is a pullback.

2) Suppose $\Psi(\hat{\mathscr{D}})$ is a pullback diagram in $\mathbb{C}$. Since $\Psi(\mathscr{D})$ is a pullback diagram in $\mathbb{C}$. as well, $\Psi(\hat{\mathscr{D}}) \cong \Psi(\mathscr{D})$. But then $\hat{\mathscr{D}} \cong \mathscr{D}$ since Ψ reflects isomorphisms. Since $\mathscr{D}$ is a pullback, so is $\hat{\mathscr{D}}$.

4.4.4 Ψ preserves and reflects monomorphisms:

a map $(A, a) \xrightarrow{f} (B, b)$ is a monomorphism (in $\mathbb{C}/X$) iff $A \xrightarrow{f} B$ is a monomorphism (in $\mathbb{C}$).

This can be checked directly (of course), but it also follows from 4.3 and the fact that, in any category $\mathbb{C}$, $A \xrightarrow{f} B$ is a monomorphism iff

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1_A}\uparrow & & \uparrow{\scriptstyle f} \\ A & \xrightarrow[1_A]{} & A \end{array}$$

is a pullback diagram (check!).

4.4.5 Ψ preserves and reflects extremal epi's:

an arrow $(A, a) \xrightarrow{f} (B, b)$ is e.e. iff $f : A \to B$ is e.e.

proof: Complete: $(A, a) \xrightarrow{f} (B, b)$.

1) Suppose $A \xrightarrow{f} B$ (e.e. in $\mathbb{C}$), to show $\overline{}$

$(A, a) \underset{?}{\overset{f}{\twoheadrightarrow}} (B, b)$. (*)

Let:

$$\begin{tikzcd} (A,a) \arrow[rr, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, shift right=4ex, "="] & & (B,b) \\ & (C,c) \arrow[ur, tail, "m"'] & \end{tikzcd}$$

(mono)

to show m is an iso (?). By 4.4, we have

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, shift right=4ex, "="] & & B \\ & C \arrow[ur, tail, "m"'] & \end{tikzcd}$$

, $\underline{m \text{ mono}}$,

in $\mathbb{C}$. Therefore, $m: C \to B$ is an iso; by 4.4, $m: (C,c) \longrightarrow (B,b)$ is an iso. This shows $(*)$, p 29.

2) Suppose $(A,a) \overset{f}{\twoheadrightarrow} (B,b)$, to show

$$A \underset{?}{\overset{f}{\twoheadrightarrow}} B$$

[actually, this is the more surprising part, since $\mathbb{C}/X = \mathbb{D}$ may suffer "collapse"; see also later].

To this end, let:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, shift right=4ex, "="] & & B \\ & C \arrow[ur, tail, "m"'] & \end{tikzcd}$$

m mono; in $\mathbb{C}$.

Consider:

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "g"'] \arrow[dr, "a"] & & B \arrow[dl, "b"'] \\
{} \arrow[r, phantom, "2?"] & X & {} \arrow[l, phantom, "1?"] \\
C \arrow[ur, "c?"] \arrow[rr, equal] & & C \arrow[ul, "c"'] \arrow[uu, tail, "m"']
\end{tikzcd}
$$

Define : $c \overset{\text{def}}{=} bm$; 1?: OK. Need: 2?:

$$
\begin{array}{ccc}
cg & \overset{?}{=} a & = bf \\
\| & & \| \checkmark \\
(bm)g & & \\
\| & & \\
b(mg) & = & bf
\end{array}
$$

By 4.4: $m : (C, c) \longrightarrow (B, b)$ is a mono

I have (*), bp p. 30, in $\mathbb{D}$. Hence,

$m : (C, c) \longrightarrow (B, b)$ is an iso. Hence;

$m : C \longrightarrow B$ is an iso.

$\boxed{\text{Done}}$

4.(5 ⊙) $\mathbb{D} = \mathbb{C}/X$ satisfies 1.2) ; page 1.

Suppose $(A,a) \xrightarrow{f} (B,b)$. Factor $A \xrightarrow{f} B$ in $\mathbb{C}$, according to 1.2):

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "g"'] & \arrow[d, phantom, "="] & B \\ & C \arrow[ur, tail, "m"'] & \end{tikzcd}$$

Define $c : C \xrightarrow{c} X$ so that the following commutes:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dd, "g"'] \arrow[dr, "a"] & & B \arrow[dl, "b"'] \\ & X & \\ C \arrow[ur, "c?"] \arrow[rr, equal] & & C \arrow[ul, "c?"'] \arrow[uu, "m"'] \end{tikzcd}$$

This is done as on the previous page. We have

$$\begin{tikzcd} (A,a) \arrow[rr, "f"] \arrow[dr, "g"'] & \arrow[d, phantom, "="] & (B,b) \\ & (C,c) \arrow[ur, "m"'] & \end{tikzcd}$$

in $\mathbb{C}/X$, and by 4.4 and 4.5, m is a mono (in $\mathbb{D}$) and g is e.e. (in $\mathbb{D}$). $\boxed{\text{Done}}$

4.(6°) Condition 1.3) for $\mathbb{C}/X$: stability of 'e.e.' under pullback: immediate from the preservation and reflection properties of $\Psi: \mathbb{C}/X \longrightarrow \Psi$ discussed above.

4.(7°) <u>Subobject lattices:</u> before subobjects, consider mere monomorphisms. In any category $\mathbb{C}$ and object Y, let Mono(Y) denote the class of all monomorphisms $A \overset{\hat{a}}{\rightarrowtail} Y$ into Y. Mono(Y) is a pre-order: we have the relation

$$(A, \hat{a}) \leq (B, \hat{b}) \iff \text{there is (nec. unique)}$$

$$f: A \to B \quad \text{s.t.} \quad \hat{b} f = \hat{a}$$

$$\begin{tikzcd}
A \arrow[rr, tail, "\hat{a}"] \arrow[dr, dashed, "f"'] & \arrow[d, phantom, "\circleddash"] & Y \\
& B \arrow[ur, tail, "\hat{b}"'] &
\end{tikzcd}$$

$\leq$ is reflexive and transitive, but, of course, it may fail to satisfy antisymmetry — the notion of subobject "corrects" this circumstance. But, in fact, the pre-order Mono(Y) is 'more basic' than Sub(Y), and can be used instead of $\mathrm{Sub}(Y)$. The reason I bring it up here is that, in our situation, we have a sharp relationship connecting Mono-preorders, and the corresponding, weaker, Sub-related fact is a consequence.

Consider $\Psi : \mathbb{C}/X \to \mathbb{C}$ as before. Let $(Y, y) \in \mathrm{Ob}(\mathbb{C}/X)$. Then, since Ψ preserves monomorphisms (see: 4.4, p29), Ψ induces a mapping

$$\Psi_{(Y,y)} : \mathrm{Mono}_{\mathbb{D}}((Y,y)) \longrightarrow \mathrm{Mono}_{\mathbb{C}}(Y)$$

$$(\mathbb{D} = \mathbb{C}/X)$$

which, since Ψ is faithful, is one-to-one. But in fact, $\Psi_{(Y,y)}$ is a bijection: if $A \overset{\hat{a}}{\rightarrowtail} Y$, then, for $a = y\hat{a}$, $((A,a),\hat{a}) \in \mathrm{Mono}(Y,y)$:

$$\begin{tikzcd} A \arrow[rr, tail, "\hat{a}"] \arrow[dr, "a"'] & {} \arrow[d, phantom, "\circlearrowleft"] & Y \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

Moreover, $\Psi_{(Y,y)}$ is, obviously, an <u>isomorphism</u> of the preorders involved; therefore, an isomorphism

$$\Psi_{(Y,y)} : \mathrm{Sub}_{\mathbb{D}}((Y,y)) \xrightarrow{\ \cong\ } \mathrm{Sub}_{\mathbb{C}}(Y)$$

is induced. It immediately follows that condition 1.4 holds for $\mathbb{D} = \mathbb{C}/X$: subobjects form lattices. Moreover, the isomorphisms above are compatible with the pullback functors: for $(Z,z) \xrightarrow{f} (Y,y)$ in $\mathbb{C}/X$; the following diag:

$$\begin{tikzcd}
\mathrm{Sub}_{\mathbb{D}}((Y,y)) \arrow[r, "\Psi_{(Y,y)}", "\cong"'] \arrow[d, "f^*"'] \arrow[dr, phantom, "="] & \mathrm{Sub}_{\mathbb{C}}(Y) \arrow[d, "f^*"] \\
\mathrm{Sub}_{\mathbb{D}}((Z,z)) \arrow[r, "\Psi_{(Z,z)}"'] & \mathrm{Sub}_{\mathbb{C}}(Z)
\end{tikzcd}$$

commutes; as a consequence, condition 1.5 follows too.

We have proved that $\mathbb{C}/X$ is a coherent category. We observe that we use

4.7 We check that

$$\underline{\Phi} = \underline{\Phi}_X : \mathbb{C} \longrightarrow \mathbb{C}/X$$

is a coherent functor.

In 4.2 (p 23), we saw that $\underline{\Phi}$ preserves the terminal object, and it was mentioned that it preserves pullbacks, for a "general reason". But, here is a direct view:

At issue:

$$\begin{tikzcd} A \arrow[r, "f"] & C \\ P \arrow[u, "\pi_0"] \arrow[r, "\pi_1"'] \arrow[ur, phantom, "\mathrm{pb}"] & B \arrow[u, "g"'] \end{tikzcd} \quad \overset{??}{\Longrightarrow} \quad \begin{tikzcd} A\times X \arrow[r, "f\times X"] & C\times X \\ P\times X \arrow[u, "\pi_0\times X"] \arrow[r, "\pi_1\times X"'] \arrow[ur, phantom, "\mathrm{pb}"] & B\times X \arrow[u, "g\times X"'] \end{tikzcd}$$

we have already reduced the problem into something in $\mathbb{C}$ alone. For an arbitrary object Y, arrows $\langle h, k\rangle : Y \xrightarrow{i} Z\times X$ for any Z (there are four candidates for Z: P, A, B, C) are in bijective correspondence with pairs of arrows $(Y \xrightarrow{h} Z, Y \xrightarrow{k} X)$. For $Y \xrightarrow{i_A} A\times X$, $Y \xrightarrow{i_B} B\times X$, the commutativity condition $(f\times X)i_A = (g\times X)i_B$ is equivalent to $fh_A = gh_B$ — the k's do not contribute.

I stop here.

4.8.1 $\underline{\Phi \text{ preserves e.e. morphisms:}}$

if $A \overset{f}{\twoheadrightarrow} B$ is e.e. in $\mathbb{C}$, then

$\Phi(A) \xrightarrow{\Phi(f)} \Phi(B)$ is e.e. in $\mathbb{C}/X$.

But the latter is $f \times X : (A, \pi_1) \longrightarrow (B, \pi_1)$:

$$\begin{tikzcd}
A \times X \arrow[rr, "f \times X"] \arrow[dr, "\pi_1"'] & & B \times X \arrow[dl, "\pi_1"] \\
& X &
\end{tikzcd}$$

and we know (4.4.5, p 29) that

$f \times X : (A, \pi_1) \longrightarrow (B, \pi_1)$ is e.e. in $\mathbb{C}/X$

iff $f \times X : A \times X \longrightarrow B \times X$ is e.e. in $\mathbb{C}$.

We have that

$$\begin{tikzcd}
A \times X \arrow[r, "f \times X"] \arrow[d, "\pi_0"'] & B \times X \arrow[d, "\pi_0"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

is a pullback diagram (in $\mathbb{C}$) (check this general fact!). By property 1.3) - stability of the pullback 'e.e.' property — , we have that $f \times X$ is e.e as desired.

(4.8.2) : The map induced by Φ:

$$\mathrm{Sub}_{\mathbb{C}}(Y) \xrightarrow{\Phi} \mathrm{Sub}_{\mathbb{C}/X}(\Phi(Y))$$

is, by the isomorphism

$$\mathrm{Sub}_{\mathbb{C}/X}((Y \times X, \pi_1)) \xrightarrow{\simeq} \mathrm{Sub}_{\mathbb{C}}(Y \times X)$$

is essentially the same as

$$\mathrm{Sub}_{\mathbb{C}}(Y) \xrightarrow{f^*} \mathrm{Sub}_{\mathbb{C}}(Y \times X)$$

for the product projection $f = \pi_0 : Y \times X \to Y$ (using (again) that

$$\begin{tikzcd}
A \times X \arrow[r, "a \times X"] \arrow[d, "\pi_0"'] & Y \times X \arrow[d, "\pi_0"] \\
A \arrow[r, "a"'] & Y
\end{tikzcd}$$

is a pullback diagram.) This shows that $\Phi : \mathbb{C} \to \mathbb{C}/X$ has the preservation properties for the lattice structure (see p. 8) page 8 — as a consequence of 1.5)

(p. 4) for $\mathbb{C}$.

This completes the proof of the Proposition, p 22

<u>Remark</u>: One sees the <u>active</u> role of the pullback stability properties 1.3) and 1.5) <u>first</u> in the preservation properties of $\Phi_X : \mathbb{C} \longrightarrow \mathbb{C}/X$.