

Bits and pieces.

(incomplete! see later...)

March 2 / 015

$\mathbb{C}$: coherent category

1

1. Definitions

1.1 $\mathbb{C}$ is complete if_{def} the top element of the subobject lattice of $1_{\mathbb{C}}$ (the terminal object of $\mathbb{C}$) is (finitely) irreducible:

whenever I is a finite set (possibly empty), $U_i \in \text{Sub}(1_{\mathbb{C}})$ ($i \in I$), and

$$\bigvee_{i \in I} U_i = T_{1_{\mathbb{C}}} \text{ (written } \top \text{ from now on)}$$

then: there is $i \in I$ such that $U_i = \top$

NB: $I = \emptyset$ and $I = 2$: sufficient.

1.2 $\mathbb{C}$ has enough global elements

(or, more simply, $\mathbb{C}$ is saturated) if_{def} for all $X \in \text{Ob}(\mathbb{C})$, if $X \xrightarrow{!X} 1_{\mathbb{C}}$ is surjective (= e.e. = extremal epi), then there is $x: 1_{\mathbb{C}} \rightarrow X$.

(at least one; usually, of course, many!)

NB: If $x: \mathbb{1}_{\mathcal{C}} \rightarrow X$, then
 $!X$ is surjective: the condition is
 necessary:

(*) for this, note that if

$$\begin{array}{ccc}
 & \xleftarrow{u} & \\
 U & \xrightarrow[m \text{ (mono)}]{} & \mathbb{1} \quad (= \mathbb{1}_{\mathcal{C}})
 \end{array}$$

then m is an isomorphism, $u = m^{-1}$;

of course, $m u = \mathbb{1}_{\mathbb{1}}$

but also, $u m = \mathbb{1}_U$

since $m u m = m \mathbb{1}_U = \mathbb{1}_{\mathbb{1}}$ ($\mathbb{1}$ is terminal)

and m is a mono

now: take $X \xrightarrow{!X} \mathbb{1}$

and factor:

$$\begin{array}{ccc}
 & & \nearrow m \\
 X & \searrow e & U
 \end{array}$$

With $x: \mathbb{1} \rightarrow X$, I have $u: \mathbb{1} \rightarrow U$

with $u = \underset{\text{def}}{e \circ !X}$. Thus, m is an isomorphism,

and $!X$ is surjective, since e is.

(2). CLAIM: suppose ① complete (as in (1.1))
and:

$$x: \mathbb{1} \rightarrow X$$

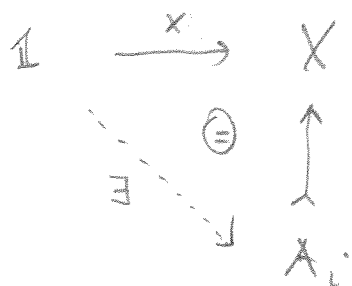
$$\text{and } \bigvee_{i \in I} A_i = T_X \quad \text{in } \text{Sub}(X).$$

↑
finite

Then there is $i \in I$ such that

$$x \in A_i$$

meaning



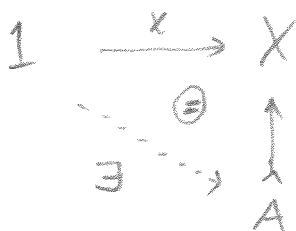
Better (of course, the same): "same" hyp's:

$$x \in \bigvee_{i \in I} A_i \Rightarrow \exists i \in I. x \in A_i$$

$x: \mathbb{1} \rightarrow X$
 $A_i \in \text{Sub}(X)$
 $I: \text{finite}$
 $\mathbb{C}: \text{complete}$

here, of course, $x \in A$ ($x: \mathbb{1} \rightarrow X, A \in \text{Sub}(X)$)

means

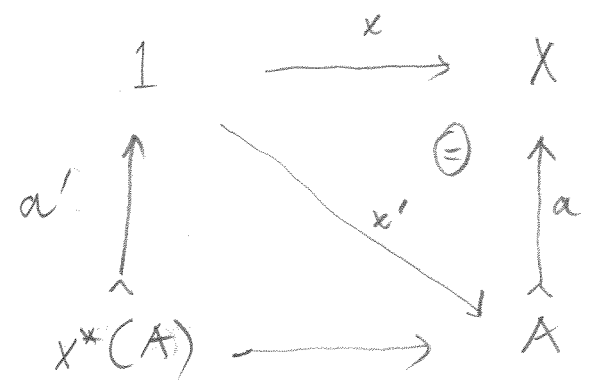


For this, 1st note:

$$\begin{cases} A \in \text{Sub}(X) \\ 1 \xrightarrow{x} X \end{cases}$$

"If $x \in A$, then $A(x)$ is true"

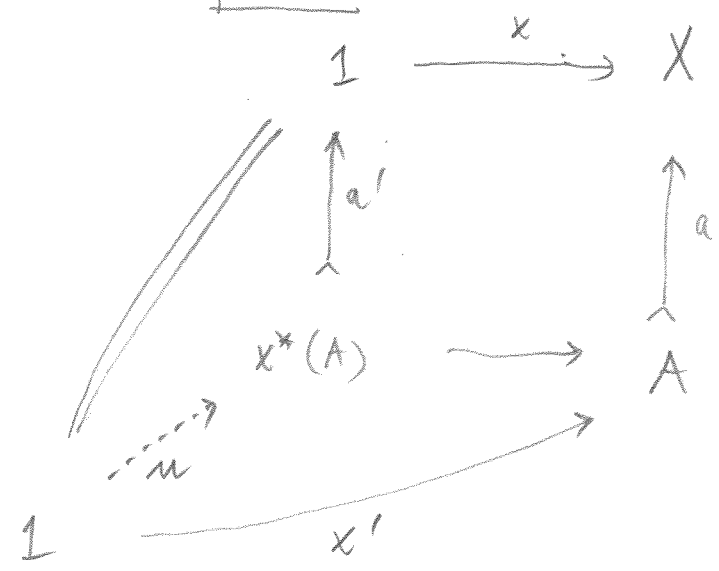
meaning:



square: pullback
 $ax' = x$

$\Rightarrow a'$ is an isomorphism.

because: we pull back:



now, use $(*)$, p 2

Return to box, p 3:

5

$$\begin{array}{ccc}
 1 & \xrightarrow{x} & X \\
 \uparrow a' & \searrow x' & \uparrow a \\
 x^*(\bigvee_{i \in I} A_i) & \xrightarrow{\quad} & \bigvee_{i \in I} A_i
 \end{array}
 \quad \text{with } x' = a \circ a'$$

$\therefore a'$ is an isomorphism; or more simply,

$$x^*(\bigvee_{i \in I} A_i) = \top \quad (\text{equality in } \text{Sub}(1_C)).$$

$$\text{But: } x^*(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} x^*(A_i)$$

(property of $\mathbb{C}$). By completeness,

$$x^*(A_i) = \top.$$

Now:

$$\begin{array}{ccc}
 1 & \xrightarrow{x} & X \\
 \uparrow a' \cong \uparrow (a')^{-1} & & \uparrow a \\
 x^*(A_i) & \xrightarrow{p} & A_i
 \end{array}$$

$$\text{have: } 1 \xrightarrow{p \circ (a')^{-1}} A_i$$

And:

$$\begin{array}{ccc}
 1 & \xrightarrow{x} & X \\
 & \searrow \textcircled{=} & \uparrow a \\
 & & A_i
 \end{array}
 \quad ?$$

$$a \cdot p(a')^{-1} \stackrel{?}{=} x$$

$$\Leftrightarrow a \cdot p = x \cdot a', \text{ and that is true.}$$

This shows: $x \in A_i$

- box on p. 3 proved.

③. Consider the global-sections functor

$$\Gamma = \mathbb{C}(1_{\mathbb{C}}, -) : \mathbb{C} \longrightarrow \text{Set}$$

$$X \longmapsto \mathbb{C}(1_{\mathbb{C}}, X)$$

$$= \{x \mid x: 1 \longrightarrow X\}$$

As any representable functor, Γ preserves all existing limits in $\mathbb{C}$.

For every X , Γ induces

$$\text{Sub}_{\text{①}}(X) \xrightarrow{\Gamma_X} \mathcal{P}(\Gamma(X)) \\ = \text{Sub}_{\text{Set}}(\Gamma(X))$$

$$A \xrightarrow{a} X \quad | \quad \xrightarrow{\quad} \Gamma(A) \xrightarrow{\Gamma(a)} \Gamma(X) \\ \text{(identified with subset } \text{Im}(\Gamma(a)) \text{ of } \Gamma(X))$$

(since Γ is left exact (preserves finite limits),
it preserves monomorphisms)

and, of course, if (A, a) and (A', a') define
the same subobject, so do $(\Gamma(A), \Gamma(a))$ and
 $(\Gamma(A'), \Gamma(a'))$. Γ_X is a morphism of partial
orders; Γ_X is a morphism of $(T, \wedge)$ -semilattices.

But now:

{ if $\mathbb{C}$ is complete, Γ_X is a morphism of
lattices: Γ_X preserves finite sup's

this is Box, p 3.