Bits and pieces. (incomplete! see later...)

$\mathbb{C}$: coherent category

1. Definitions

1.1 $\mathbb{C}$ is complete if_{def} the top element of the subobject lattice of $1_{\mathbb{C}}$ (the terminal object of $\mathbb{C}$) is (finitely) irreducible:

whenever I is a finite set (possibly empty), $U_i \in \mathrm{Sub}(1_{\mathbb{C}})$ $(i \in I)$, and

$\bigvee_{i\in I} U_i = \top_{1_{\mathbb{C}}}$ (written $\mathbb{T}$ from now on)

then: there is $i \in I$ such that $U_i = \mathbb{T}$

NB: $I = \emptyset$ and $I = 2$: sufficient.

1.2 $\mathbb{C}$ has enough global elements (or, more simply, $\mathbb{C}$ is saturated) if_{def} for all $X \in \mathrm{Ob}(\mathbb{C})$, if $X \xrightarrow{!_X} 1_{\mathbb{C}}$ is surjective (= e.e. = extremal epi), then there is $x : 1_{\mathbb{C}} \to X$.

(at least one; usually, of course, many!)

<u>NB</u>: If $x : \mathbb{1}_{\mathbb{C}} \to X$, then

$!_X$ is surjective: the condition is

necessary;

(*) for this, note that if

$$\begin{tikzcd} U \arrow[r, tail, "m"'] & \mathbb{1} \arrow[l, bend right, "u"'] \end{tikzcd} \quad (= \mathbb{1}_{\mathbb{C}})$$

(mono)

then m is an isomorphism, $u = m^{-1}$;

of course, $mu = 1_{\mathbb{1}}$

but also, $um = 1_U$

since $mum = m1_U = 1_{\mathbb{1}}$ ($\mathbb{1}$ is terminal)

and m is a mono

now: take $X \xrightarrow{!_X} \mathbb{1}$

and factor:

$$\begin{tikzcd} X \arrow[rr, "!_X"] \arrow[dr, two heads, "e"'] & & \mathbb{1} \\ & U \arrow[ur, tail, "m"'] & \end{tikzcd}$$

With $x : \mathbb{1} \to X$, I have $u : \mathbb{1} \to U$

with $u \underset{\text{def}}{=} e \circ !_X$. Thus, m is an isomorphism,

and $!_X$ is surjective, since e is.

2. CLAIM: suppose $\mathbb{C}$ complete (as in 1.1)

and:

$$x : \mathbb{1} \longrightarrow X$$

and $\bigvee_{i \in I} A_i = \top_X$ in $\mathrm{Sub}(X)$.

finite

Then there is $i \in I$ such that $\boxed{x \in A_i}$

meaning

$$\begin{tikzcd} \mathbb{1} \arrow[r, "x"] \arrow[dr, dashed, "\exists"'] & X \arrow[d, phantom, "="] \\ & A_i \arrow[u, tail] \end{tikzcd}$$

Better (of course, the same): "same" hyp's:

$x : \mathbb{1} \to X$
$A_i \in \mathrm{Sub}(X)$
I: finite
$\mathbb{C}$: complete

$$\boxed{x \in \bigvee_{i \in I} A_i \Rightarrow \exists i \in I \,.\, x \in A_i}$$

here, of course, $x \in A$ $\quad (x : \mathbb{1} \to X,\ A \in \mathrm{Sub}(X))$

means

$$\begin{tikzcd} 1 \arrow[r, "x"] \arrow[dr, dashed, "\exists"'] & X \arrow[d, phantom, "="] \\ & A \arrow[u, tail] \end{tikzcd}$$

For this, 1st note:

$$\begin{cases} A \in \mathrm{Sub}(X) \\ 1 \xrightarrow{x} X \end{cases}$$

"If $x \in A$, then $A(x)$ is true"

meaning:

$$\begin{tikzcd} 1 \arrow[r, "x"] \arrow[dr, "x'"] & X \\ x^*(A) \arrow[u, tail, "a'"] \arrow[r] & A \arrow[u, tail, "a"'] \end{tikzcd}$$

square: pullback
$ax' = x$

$\Rightarrow$ a' is an isomorphism.

because: use pullback:

$$\begin{tikzcd} & 1 \arrow[r, "x"] & X \\ & x^*(A) \arrow[u, tail, "a'"'] \arrow[r] & A \arrow[u, tail, "a"'] \\ 1 \arrow[uur, equal] \arrow[ur, dashed, "u"'] \arrow[urr, bend right, "x'"'] & & \end{tikzcd}$$

now, use $(*)$, p $\boxed{2}$

Return to box, p $\boxed{3}$:

$$\begin{tikzcd}
1 \arrow[r, "x"] \arrow[drr, "x'"] & \arrow[d, phantom, "="] & X \\
x^*\left(\bigvee\limits_{i\in I} A_i\right) \arrow[u, tail, "a'"] \arrow[rr] & & \bigvee\limits_{i\in I} A_i \arrow[u, tail, "a"']
\end{tikzcd}$$

$\therefore$ a' is an isomorphism; or more simply,

$$x^*\left(\bigvee_{i\in I} A_i\right) = \mathbb{T} \quad \text{(equality in } \mathrm{Sub}(\mathbb{1}_{\mathbb{C}})\text{)}.$$

But: $x^*\left(\bigvee\limits_{i\in I} A_i\right) = \bigvee\limits_{i\in I} x^*(A_i)$

(property of $\mathbb{C}$). By completeness,

$$x^*(A_i) = \mathbb{T}.$$

Now:

$$\begin{tikzcd}
1 \arrow[r, "x"] \arrow[d, bend left, "(a')^{-1}"] & X \\
x^*(A_i) \arrow[u, "a' \cong"] \arrow[r, "p"'] & A_i \arrow[u, tail, "a"']
\end{tikzcd}$$

have: $1 \xrightarrow{p\circ(a')^{-1}} A_i$

And:

$$\begin{tikzcd} 1 \arrow[r, "x"] \arrow[dr, "p\cdot(a')^{-1}"'] & X \\ & A_i \arrow[u, hook, "a"'] \end{tikzcd} \quad ?$$

$$a\, p\, (a')^{-1} \overset{?}{=} x$$

$$\Longleftrightarrow$$

$$a\, p = x\, a' \,,$$ and that is true.

This shows: $\underline{x \in A_i}$

– box on p. $\boxed{3}$ proved.

(3.) Consider the global-sections functor

$$\Gamma = \mathbb{C}(1_{\mathbb{C}}, -) : \mathbb{C} \longrightarrow \mathrm{Set}$$

$$X \longmapsto \mathbb{C}(1_{\mathbb{C}}, X)$$

$$= \{ x \mid x : 1 \longrightarrow X \}$$

As any representable functor, Γ preserves all existing limits in $\mathbb{C}$.

For every X, Γ induces

$$\mathrm{Sub}_{\mathbb{C}}(X) \xrightarrow[\Gamma_X]{} \mathcal{P}(\Gamma(X)) = \mathrm{Sub}_{\mathrm{Set}}(\Gamma(X))$$

$$A \overset{a}{\rightarrowtail} X \longmapsto \Gamma(A) \overset{\Gamma(a)}{\rightarrowtail} \Gamma(X)$$

(identified with subset $\mathrm{Im}(\Gamma(a))$ of $\Gamma(X)$)

(since Γ is left exact (preserves finite limits), it preserves monomorphisms)

and, of course, if (A, a) and (A', a') define the same subobject, so do $(\Gamma(A), \Gamma(a))$ and $(\Gamma(A'), \Gamma(a'))$. Γ_X is a morphism of partial order; Γ_X is a morphism of $(T, \wedge)$-semilattices.

But now:

if $\mathbb{C}$ is complete, Γ_X is a <u>morphism of</u> lattices: <u>Γ_X preserves finite sup's</u>

this is $\boxed{\text{Box}}$, p $\boxed{3}$.