April 14/2015

$\boxed{\text{SKETCHES}}$ (continued)

$\boxed{4}$ sketch-specifying the coherent doctrine (continued) (previously: $\boxed{31}$ – $\boxed{41}$)

now: $\boxed{42}$ — $\boxed{69}$

14 Sketch-specifying the coherent doctrine (continued)

$\boxed{Ax_{12}}$ ::

Ax_{12} :

$$\begin{tikzcd} 0 \arrow[r, "4"] & 2 \\ & 1 \arrow[u, "5"'] \end{tikzcd} \longrightarrow \underbrace{\begin{tikzcd} 0 \arrow[r, "4"] & 2 \\ 3 \arrow[u, "6"] \arrow[r, "7"'] \arrow[ur, phantom, "pb"] & 1 \arrow[u, "5"'] \end{tikzcd}}_{A_1}$$

$K_{pb}(A_1) = \{ id_{\bar{K}_{pb}} \}$

see p. 34

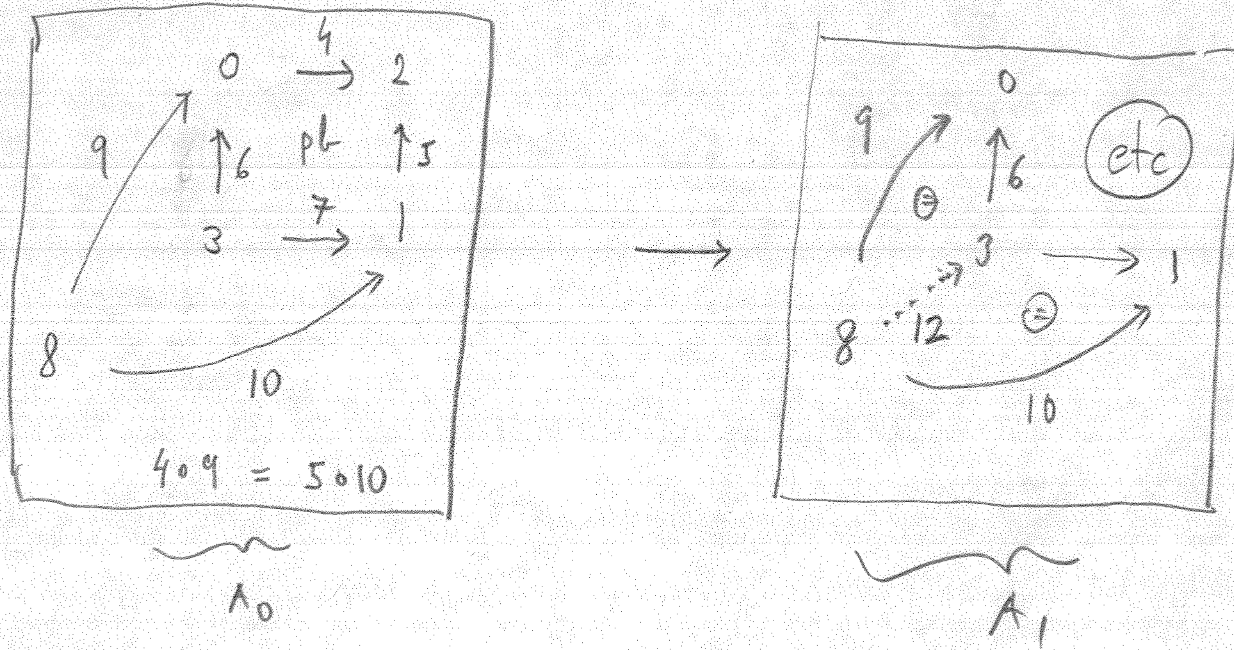
$\boxed{Ax_{13},\ Ax_{14},\ Ax_{15}}$: the analogs of Ax_9, Ax_{10}, A_{11} respectively.

For instance

$Ax_{13} = Ax_{pb\text{-}univ\text{-}ex}$ = axiom for the existence part of the universal property of the pullback

looks like this:

Ax_{13}:



$|A_0|$ has the arrow,
in addition to the ones shown,
also $8 \xrightarrow{11} 2$

$$K_{comm}(A_0) = \left\{ \begin{tikzcd} 8 \arrow[r, "9"] \arrow[rr, bend right, "11"'] & 0 \arrow[r, "4"] & 2 \end{tikzcd}, \quad \begin{tikzcd} 8 \arrow[rr] \arrow[dr, "10"'] & & 2 \\ & 1 \arrow[ur, "5"'] & \end{tikzcd} \right\}$$

$$K_{pb}(A_0) = \left\{ \begin{tikzcd} 0 \arrow[r, "4"] & 2 \\ 3 \arrow[u, "6"] \arrow[r, "7"'] & 1 \arrow[u, "5"'] \end{tikzcd} \right\}$$

Note that constructions of sketches can be expressed categorically, by using pushouts. For instance, A_1 in this case (the target) can be expressed by the following pushout:

$$\begin{tikzcd} A_0 \arrow[r, "Ax_{14}"] & A_1 \\ {} \arrow[u, "\text{incl.}"] \arrow[ur, phantom, "\text{p.o.}"] & {} \arrow[u, "\text{incl}"'] \end{tikzcd}$$

$$\begin{tikzcd} & 0 & \\ & 3 \arrow[u, "6"] \arrow[r, "7"] & 1 \\ 8 \arrow[uur, "9"] \arrow[urr, "10"'] & & \end{tikzcd} \longrightarrow \begin{tikzcd} & 0 & \\ & 3 \arrow[u, "6"] \arrow[r, "7"] & 1 \\ 8 \arrow[uur, "9"] \arrow[ur] \arrow[urr, "10"'] & & \end{tikzcd}$$

$\boxed{Ax_{15}}$ ::

$$\underbrace{\begin{tikzcd} 0 \arrow[dr, "4"] & & 8 \arrow[ll, "9"'] \arrow[dd, "10"] \\ & 2 & \\ 3 \arrow[uu, "6"] \arrow[rr, "7"'] & & 1 \arrow[ul, "5"'] \end{tikzcd} \quad 3 \xrightarrow[11]{\cong} 8}_{A_0} \longrightarrow \underbrace{\text{add the map } \begin{pmatrix} 0\,1\,2\,3\,4\,5\,6\,7 \\ 0\,1\,2\,\underline{8}\,4\,5\,\underline{9}\,\underline{10} \end{pmatrix} \text{ to } K_{pb}(A_1)}_{A_1}$$

$K_{pb}(A_0) = \{\text{incl} : \overline{K}_{pb} \to |A_0|\}$

(thus: $\begin{tikzcd} 0 \arrow[dr, "4"'] & & 8 \arrow[ll, "9"'] \arrow[dd, "10"] \\ & 2 & \\ & & 1 \arrow[ul, "5"] \end{tikzcd}$)

We have completed the specification of the concept of "category with finite limits":

specification names:

$$K_{id},\ K_{comm},\ K_{inv},\ K_{term},\ K_{pb}$$

with corresponding specification templates

$$\bar{K}_{id},\ \ldots$$

in $\mathcal{G}$ = Graph; and with axioms:

$$\mathbb{R}_{fl} \stackrel{def}{=} \{Ax_0, \ldots, Ax_{15}\}.$$

We have that $\boxed{Cat_{finlim} \subset SkCat_{finlim}}$

the subcategory being specified by $\mathbb{R}_{fl}$.

— · —

To the rest of the specification of coherent categories:

$$\begin{cases} K_{mono} : \text{ new spec name (for "monomorphism")} \\ \bar{K}_{mono} :=: \boxed{0 \xrightarrow{2} 1} \quad (\in Ob(Graph)) \end{cases}$$

$$\begin{cases} K_{ee} = K_{extremal\ epi} : \text{ new spec name (for ee's)} \\ \bar{K}_{ee} :=: \boxed{0 \xrightarrow{2} 1} \end{cases}$$

S | 46

Axioms:

$\boxed{Ax_{16} \ \& \ Ax_{17}}$: expressing :

$$X \xrightarrow{f} Y \quad \text{is a monomorphism}$$

$$\Longleftrightarrow$$

$$\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[d, "id_X"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

is a pullback diagram

(compare : Ax_6 & Ax_7 for the notion of isomorphism)

$\boxed{Ax_{18}}$ :: axiom for the definition of "extremal epi" :

Ax_{18} :

$$\underbrace{\boxed{\begin{tikzcd}
0 \arrow[rr, two heads, "2 \ (ee)"] \arrow[dr, "4"'] & \arrow[d, phantom, "="] & 1 \\
& 3 \arrow[ur, tail, "5"'] \arrow[ur, phantom, "(mono)", shift right=4ex] &
\end{tikzcd}}}_{A_0}
\longrightarrow
\underbrace{\boxed{\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\cong"] & 1 \\
& 3 \arrow[ur, "(iso)"'] &
\end{tikzcd} \\ (etc)}}_{A_1}$$

[A more intelligent description of Ax_{18} is as follows. Granting A_0 first, A_1 is obtained as the pushout

$$
\begin{tikzcd}
\underbrace{\boxed{0 \xrightarrow{2} 1}}_{U} \arrow[r] \arrow[d] & \underbrace{\boxed{0 \xrightarrow[\cong]{2} 1}}_{V} \arrow[d] \\
A_0 \arrow[r] & A_1 \arrow[ul, phantom, "\ulcorner \text{p.o.}"]
\end{tikzcd}
$$

Of course, V has $K_{iso}(V) = \{ id_{\overline{K}_{iso}} \}$.

(A_0 itself can be "built" from smaller ingredients:

let $Arr :=: \boxed{0 \xrightarrow{2} 1}$ (a graph),

$$a_0 = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 1 & 5 \end{pmatrix} : Arr \longrightarrow \overline{K}_{comm}$$

$$a_1 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} : Arr \longrightarrow \overline{K}_{comm}$$

Then A_0 is a pushout:

$$
\begin{tikzcd}
Arr \sqcup Arr \arrow[r, "j_0 \sqcup j_1"] \arrow[d, "{[a_0, a_1]}"'] & \overline{K}_{ee} \sqcup \overline{K}_{mono} \arrow[d] \\
\overline{K}_{comm} \arrow[r] & A_0
\end{tikzcd}
$$

)]

$\boxed{Ax_{19}}$: axiom for factorization:

A_{19} : $\boxed{0 \xrightarrow{2} 1} \longrightarrow$
$$\boxed{\begin{tikzcd} 0 \arrow[rr, "2"] \arrow[dr, "4"'] & \arrow[d, phantom, "\ominus"] & 1 \\ & 3 \arrow[ur, "5"'] & \end{tikzcd}}$$
4 : extremal epi
5 : mono

$\boxed{Ax_{20}}$: stability of (e.e.) under pullback

A_{20} :
$$\boxed{\begin{tikzcd} 0 \arrow[r, two heads, "2"] \arrow[dr, phantom, "\mathrm{pb}"] & 1 \\ 3 \arrow[u] \arrow[r, "5"'] & 4 \arrow[u] \end{tikzcd}}$$
2 : ee

$\longrightarrow$ $\boxed{\text{add : (5 is e.e.)}}$

— · —

$\mathbb{R}_{reg} \overset{def}{=} \{Ax_0, \ldots, Ax_{20}\}$: specification of "regular category"

within "sketch category for regular category" : SkReg

sk_{reg} : $\mathrm{Reg} \underset{\text{full embedding}}{\overrightarrow{\subset}} \mathrm{SkReg}$

NB: 'regular category' is defined by

conditions 1.1), 1.2) & 1.3) on page 1

in the notes "Coherent completeness" –

in other words, ignore the references (in 1.4) and 1.5))

to sup's of subobjects in loc. cit. The notion

of a 'regular functor', a morphism in Reg, is

obtained similarly.

$$\mathrm{Sk\,Reg} \overset{\mathrm{def}}{=} \mathrm{Sk}(\mathrm{Graph}, \mathbb{K}_{reg}, \underbrace{\mathbb{R}_{reg}}_{\text{see above!}})$$

where $\mathbb{K}_{reg} = \{K_{id}, K_{comm}, K_{iso}, K_{term}, K_{pb}, K_{mono}, K_{ee}\}$

The <u>proof</u> of, for $S \in \mathrm{SkReg}$,

$$\boxed{S \in \mathrm{Reg} \iff S \models \mathbb{R}_{reg}}$$

(where, of course, '$S \in \mathrm{Reg}$' means $S = sk_{reg}(\mathbb{C})$, $\mathbb{C} \in \mathrm{Reg}$, some (unique) $\mathbb{C}$)

(which is the <u>meaning</u> of saying that "$\mathbb{R}_{reg}$ specifies the notion of regular category") is of <u>some</u> interest, similarly to what was said about "category with a terminal object" above.

Here is an outline: $\boxed{\text{Assume: } \mathcal{S} \models \mathbb{R}_{reg}, \text{ to show } \mathcal{S} = sk_{reg}(\mathbb{C}) \text{ for a regular category } \mathbb{C}.}$ Take $\mathcal{S}$ restricted to finite limits: $\mathcal{S} \upharpoonright Sk_{finlim}$. We know that it <u>is</u> a category, call it $\mathbb{C}$, with finite limits.

We have: any arrow f, f is mono in $\mathbb{C}$ iff f is mono in $\mathcal{S}$. (we use Ax_{16} & Ax_{17}). We infer:

$$f \text{ is e.e. in } \mathcal{S}$$

$$\Longrightarrow f \text{ is e.e. in } \mathbb{C}.$$

To show

$$f \text{ is e.e. in } \mathbb{C}$$

$$\overset{?}{\Longrightarrow} f \text{ is e.e. in } \mathcal{S},$$

suppose: f is e.e. in $\mathbb{C}$, and factor f <u>in $\mathcal{S}$</u>:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & \arrow[d, phantom, "="] & Y \\ & Z \arrow[ur, "m"'] & \end{tikzcd}.$$

with g e.e. in $\mathcal{S}$ & m mono.

Therefore, g is e.e. in $\mathbb{C}$ too. Since f is e.e. in $\mathbb{C}$, m is an iso in $\mathbb{C}$, hence in $\mathcal{S}$ as well.

Now, we have

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "g"', "\text{e.e.}"'{xshift=-2em}] & \arrow[d, phantom, "="] & Y \\ & Z \arrow[ur, tail, "m"', "\text{iso}"{xshift=2em}] & \end{tikzcd}$$

all notions understood in the sketch $\mathcal{S}$.

The diagram

$$\begin{tikzcd} X \arrow[r, two heads, "g", "\text{ee}"'{yshift=3ex}] & Z \\ X \arrow[u, "\text{id}_X"] \arrow[r, "f"'] \arrow[ur, phantom, "\text{pb}"] & Y \arrow[u, "m"', "\cong"] \end{tikzcd}$$

is a pullback in $\mathbb{C}$ (check!) — hence, in $\mathcal{S}$ as well.

Therefore, by using A_{20}, assumed valid in $\mathcal{S}$,

<u>f is e.e. in $\mathcal{S}$.</u>

At this point, we know that all specification for $K_{id}, \dots, K_{ee}$ coincide for $\mathcal{S}$ and $\mathbb{C}$.

It is only now that we can address the proof of $\mathbb{C}$ being regular. However, knowing the agreement of the specifications of S and $sk_{regl}(\mathbb{C})$, the fact that $\mathbb{C}$ is a regular category, i.e., that conditions (1.1) 1.2) & 1.3) (page $\boxed{1}$) hold, is immediate from Ax_{19}, Ax_{20} being valid for S.

We complete the specification of 'coherent category'.

Two more specifications:

K_{bottom}, with $\overline{K}_{bottom} ::= \boxed{0 \xrightarrow{2} 1}$

K_{sup}, with

$\overline{K}_{sup}$:

$$\begin{tikzcd}
1 \arrow[dr, "6"] \arrow[drr, "4"] & & \\
& 3 \arrow[r, "8"] & 0 \\
2 \arrow[ur, "7"] \arrow[urr, "5"'] & &
\end{tikzcd}$$

$\boxed{Ax_{21}}$ ::

Ax_{21} : $\overbrace{\boxed{\overline{K}_{sup}}}^{A_0} \longrightarrow$

$$\begin{tikzcd}
1 \arrow[dr, tail, "6"'] \arrow[drr, tail, "4"] & & \\
& 3 \arrow[r, tail, "8"] \arrow[ur, phantom, "\circleddash"] \arrow[dr, phantom, "\circleddash"] & 0 \\
2 \arrow[ur, tail, "7"] \arrow[urr, tail, "5"'] & &
\end{tikzcd}$$

all arrows: monos

two triangles: commute

with

$K_{sup}(A_0)=\{id_{\overline{K}_{sup}}\}$

(of course, also $K_{sup}(\ldots)=\{id_{\overline{K}_{sup}}\}$)

$\boxed{Ax_{22}}$: existence of sup :

$$\boxed{\begin{tikzcd}
1 \arrow[dr, tail, bend left, "4"] & \\
& 0 \\
2 \arrow[ur, tail, bend right, "5"'] &
\end{tikzcd}} \longrightarrow \text{target}(Ax_{21})$$

4, 5 : mono's

$\boxed{Ax_{23}}$: universal property of sup :

$$Ax_{23}: \boxed{\begin{array}{c}
\begin{tikzcd}
1 \arrow[dr, tail, "6"'] \arrow[drr, tail, bend left, "4"] & & \\
& 3 \arrow[r, tail, "8"] \arrow[u, phantom, "="] \arrow[d, phantom, "="] & 0 \\
2 \arrow[ur, tail, "7"] \arrow[urr, tail, bend right, "5"'] & &
\end{tikzcd} \\
3 = 1 \vee 2 \\
\begin{tikzcd}
1 \arrow[dr, tail, "10"'] \arrow[drr, tail, bend left, "12"] & & \\
& 9 \arrow[r, tail, "14"] \arrow[u, phantom, "="] \arrow[d, phantom, "="] & 0 \\
2 \arrow[ur, tail, "11"] \arrow[urr, tail, bend right, "13"'] & &
\end{tikzcd}
\end{array}} \longrightarrow \boxed{\begin{tikzcd}
\text{add}: 3 \arrow[rr, "15\ (\text{new})"] \arrow[dr, "8"'] & \arrow[d, phantom, "="] & 9 \arrow[dl, "14"] \\
& 0 &
\end{tikzcd}}$$

$\boxed{Ax_{24}}$: stability of "sup" under pullback :

$\boxed{Ax_{25}}$: existence of bottom $\perp$ of lattice

$\boxed{Ax_{26}}$: universal property of $\perp$

$\boxed{Ax_{27}}$: stability of $\perp$ under pullback

} left to the reader to formulate

Proposition The category of small (or not so small) coherent categories ($\mathbf{Coh}$, resp. $\mathbf{COH}$) is specified within $\mathbf{Sk\,Coh}$ (resp., $\mathbf{Sk\,COH}$), by the axioms $\mathbb{R}_{coh} = \{Ax_0, \dots, Ax_{27}\}$.

Proof: left to the reader: follows patterns established before.

— • —

Definition Let $i : \mathcal{U} \to \mathcal{V}$ be the arrow

$$\boxed{0 \xrightarrow[2]{} 1} \longrightarrow \boxed{0 \xrightarrow[2]{\cong} 1}$$

in $\mathbf{Sk\,Coh}$. Call the sketch-entailment $\varphi : \mathbb{A} \to \mathbb{B}$ in $\mathbf{Sk\,Coh}$ a *first-order coherent entailment* if there is a pushout of the form

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\varphi"] & \mathbb{B} \\ \mathcal{U} \arrow[u, "\ulcorner\alpha\urcorner"] \arrow[r, "i"'] & \mathcal{V} \arrow[u, "\beta"'] \arrow[ul, phantom, "\lrcorner"] \end{tikzcd} \qquad \alpha \overset{def}{=} \ulcorner\alpha\urcorner(2)$$

Remarks (i) $\mathbb{B}$ differs from $\mathbb{A}$ merely by having a single new specification on an arrow of $\mathbb{A}$, declaring that arrow $\alpha \in \mathrm{Arr}(\mathbb{A})$ to be an isomorphism. For a sketch S (or a coherent category $\mathbb{C}$, $S = sk(\mathbb{C})$; think of $\mathbb{C} = \mathbf{Set}$), $S \models \varphi$ means that no matter how we give a 'model' $f : \mathbb{A} \to S$ (a sketch-morphism), the arrow $f(\alpha)$ in S is always an isomorphism.

(ii) By the very definition of 'relatively finite', (see bottom, p 29), a first-order $\varphi : A \to B$ is relatively finite. Note, however, that we insist that the sketch A (and hence B) is a small sketch. This is important because we want to apply the GCT (page 30) in which we take φ there to be an arrow of $\mathbb{S}$, assumed to be "locally finite" — maybe, I should stick to "locally finitely presentable", lfp, instead of "locally finite"; 'lfp' is the standard terminology. As we will point out: the category SkCoh is lfp. This does not prevent us from referring to the statement " Set $\models \varphi$" which takes place in SkCOH, rather than SkCoh. The definition of the concept

$$\mathbb{R} \models \varphi$$

refers to the ambient category $\mathbb{S}$: it says that for all objects S of $\mathbb{S}$, $S \models \mathbb{R}$ implies $S \models \varphi$.

(iii) We have defined the notion of deducibility, $\mathbb{R} \vdash \varphi$, "in" an almost arbitrary category $\mathbb{S}$. I want to point the fact that deducibility is uneffected by extending the ambient category $\mathbb{S}$. More precisely: <u>assume</u> that $\mathbb{S}$ is a full subcategory of $\hat{\mathbb{S}}$, both $\mathbb{S}$ and $\hat{\mathbb{S}}$ have initial objects and pushouts, and the inclusion $\mathbb{S} \hookrightarrow \hat{\mathbb{S}}$ preserves the initial object and all pushouts.

Let $\mathbb{R} \subset \mathrm{Arr}(\mathbb{S})$ and $\varphi \in \mathrm{Arr}(\mathbb{S})$. We have the separate notions

$$\mathbb{R} \underset{(\mathbb{S})}{\vdash} \varphi \qquad \text{and} \qquad \mathbb{R} \underset{(\hat{\mathbb{S}})}{\vdash} \varphi$$

according to Definition 1, p. 11 and p. 12, (where we used $\mathbb{S}$, respectively $\hat{\mathbb{S}}$, as the ambient category $\mathbb{S}$, referred to in the first sentence in section $\boxed{3}$, starting on p 11 — but used the same $\mathbb{R}$ and φ, both within the category $\mathbb{S}$.) The claim is that the two notions thus understood are equivalent: that is, for $\mathbb{R}$ and φ as stated, $\mathbb{R} \vdash_{(\mathbb{S})} \varphi$ if and only if $\mathbb{R} \vdash_{(\hat{\mathbb{S}})} \varphi$. This sounds quite trivial first; but, maybe, it is not so trivial. At any rate, it can be proved by using the Proposition on p. 14 involving the notion of deduction. With the given $\mathbb{R}$ and φ, we have the notion of deduction of φ from $\mathbb{R}$ in the sense of $\mathbb{S}$, and we have the notion of deduction of φ from $\mathbb{R}$ in the sense of $\hat{\mathbb{S}}$. It is clear that an $\mathbb{S}$-deduction of φ from $\mathbb{R}$ is also an $\hat{\mathbb{S}}$-deduction — which, using said Proposition, immediately implies that

$$\mathbb{R} \vdash_{(\mathbb{S})} \varphi \quad \Rightarrow \quad \mathbb{R} \vdash_{(\hat{\mathbb{S}})} \varphi .$$

But also, I claim, any $\hat{\mathbb{S}}$-deduction of φ from $\mathbb{R}$ (given that $\mathbb{R}$ and φ are in $\mathbb{S}$) is essentially, that is, up to isomorphism, an $\mathbb{S}$-deduction of φ from $\mathbb{R}$ — thus, if an $\hat{\mathbb{S}}$-deduction exists, so does an $\mathbb{S}$-deduction. I leave the proof of this, easy, but important fact, to the reader. Thus we see that the reverse implication

$$\mathbb{R} \underset{(\vec{\mathbb{S}})}{\vdash} \varphi \quad \Rightarrow \quad \mathbb{R} \underset{(\mathbb{S})}{\vdash} \varphi$$

also holds.

(iv) Returning to the context of remark (ii), we have $\mathbb{S} \underset{def}{=} Coh$ and $\hat{\mathbb{S}} = COH$, satisfying the conditions mentioned in (iii). Coh is the category of coherent categories $\mathbb{C}$ such that $\mathbb{C} \in \underline{U}_0$, a fixed Grothendieck universe (a universe of sets satisfying all usual set-theoretical axioms), COH is the analogous entity but for $\mathbb{C} \in \underline{U}_1$, where $\underline{U}_1$ is another Grothendieck universe, $\underline{U}_0$ being an element of $\underline{U}_1$. Set is meant — as always — the category of sets X with $X \in \underline{U}_0$. Set $\in \underline{U}_1$ automatically (maybe, not tautologically though). The present point is that, for φ an arrow in SkCoh, the category of small ($\in \underline{U}_0$) sketches, the meaning of

$$\mathbb{R}_{coh} \vdash \varphi$$

is independent of which one of the two ambient categories, Sk Coh or Sk COH, we use as the 'base'. This follows from the fact that Sk Coh is a full subcat of Sk COH, and the inclusion from the first to the second preserves (all small) colimits – and from what we said under (iii).

Note that

$$\mathbb{R}_{coh} \models \varphi$$

can also be interpreted in both Sk Coh and Sk COH, and, clearly,

$$\mathbb{R}_{coh} \underset{(\mathrm{Sk}\,\underset{\sim}{\mathrm{COH}})}{\models} \varphi \quad \Rightarrow$$

$$\mathbb{R}_{coh} \underset{(\mathrm{Sk}\,\underset{\sim}{\mathrm{Coh}})}{\models} \varphi$$

However, the GCT (see p. $\boxed{30}$) is true for both Sk Coh and Sk COH!

Both of them are locally finite precentable — Sk Coh is such relative to the universe $\underline{U}_0$

(see the definitions on pages 26 to 30 — unfortunately, the universe was not mentioned explicitly!)

and Sk COH is lfp relative to the universe $\underline{U}_1$.

Thus

$$\mathbb{R}_{coh} \underset{(Sk(\underline{OH}))}{\models} \varphi \iff \mathbb{R} \underset{(Sk\,COH)}{\vdash} \varphi$$

as well as

$$\mathbb{R}_{coh} \underset{(Sk\,C\underline{oh})}{\models} \varphi \iff \mathbb{R} \underset{(Sk\,Coh}{\vdash} \varphi$$

Since, by (iii), for φ in Sk Coh (!), the right-hand-sides are equivalent, therefore, for such φ, the left-hand sides are also equivalent: the semantic consequence relation $\mathbb{R}_{coh} \models \varphi$ is also independent of the ambient category.

We return to the 'first-order coherent entailments' of the definition on page 54.

Theorem (Gödel completeness formulated for the sketch-based syntax)

For any (small) first-order sketch-entailment φ, [see Def'n, p $\boxed{54}$], we have

$$\boxed{\begin{array}{c} \mathbf{Set} \models \varphi \\ \Longleftrightarrow \\ \text{there exists a deduction in } \mathrm{SkCoh} \\ \text{from } \underbrace{\mathbb{R}_{coh}}_{\{Ax_0, \ldots, Ax_{27}\}} \text{ of } \varphi. \\ \Longleftrightarrow \mathbb{R}_{coh} \vdash \varphi \end{array}}$$

<u>Remark</u> :

by '$\mathrm{Set} \models \varphi$'

I mean $\mathrm{sk}_{\mathrm{COH}}(\mathrm{Set}) \models \varphi$,

for the 'inclusion' functor

$$\mathrm{sk}_{\mathrm{COH}} : \mathrm{COH} \longrightarrow \mathrm{SkCOH}$$

Before I give the proof, let me point out why the statement fails for (some) φ other than "first-order" ones, in fact, for some finite sketch-entailments φ. Consider this: "given $1 \xrightarrow{\ulcorner x_0 \urcorner} X$ (1 : terminal object), and an arbitrary arrow $X \xrightarrow{f} Y$, there is $Y \xrightarrow{g} X$ such that $gf = id_X$". This is a true fact in Set: to define g, for any $y \in Y$, if $y \notin \mathrm{Im}(f) = \{f(x) : x \in X\}$, let $g(y) = x_0$, and if $y \in \mathrm{Im}(f)$, choose (axiom of choice) $x_y \in X$ s.t. $f(x_y) = y$, and put $g(y) = x_y$. This fact means that the following sketch entailment:

$$\varphi : \quad \boxed{\begin{tikzcd} 0 \arrow[dr] & 2 \\ & 1 \arrow[u, "3"'] \end{tikzcd} \quad 0 : \text{terminal}} \longrightarrow \boxed{\begin{tikzcd} 0 \arrow[dr] & 2 \arrow[d, bend left, "4"] \\ & 1 \arrow[u, "3"] \end{tikzcd} \quad 4 \circ 3 = id_1}$$

is valid in Set. However, $\mathbb{R}_{coh} \vdash \varphi$ is not true, since $\mathbb{R}_{coh} \models \varphi$ is not true — since it is not true that $Sh_{coh}(\mathbb{C}) \models \varphi$ for all coherent categories $\mathbb{C}$. Example: $\mathbb{C} = \mathrm{Set}^2$.

(Exercise: construct a very small counterexample; Set^2, i.e. $\mathrm{Set}^{\boxed{0\to 1}}$, was mentioned on p. 7, of "Gödel completeness for coherent categories").

Proof of the completeness theorem:

The tools we use are: 1) and 2):

1) Gödel completeness for coherent categories (see p. 10 of the above-mentioned notes): Given small coherent $\mathbb{C}$, arrow f in $\mathbb{C}$, then

f is an isomorphism

$\iff M(f)$ is an isomorphism

for all $M : \mathbb{C} \to \mathrm{Set}$ coherent functor

[This was proved from scratch in the above-mentioned notes]

$\boxed{2)}$ the GCT: p 30, present notes, applied to $\mathcal{S} = Sk\,Coh$.

The right-to-left direction:

$$\mathbb{R}_{coh} \vdash \varphi \quad \Rightarrow \quad \mathbb{R}_{coh} \models \varphi \quad \Rightarrow \quad Set \models \varphi$$

soundness ("trivial"; see p $\boxed{23}$; the proof is valid in Sk COH, hence applicable to sk(Set))

since $sk(Set) \models \mathbb{R}_{coh}$

For the opposite direction, <u>assume $Set \models \varphi$.</u>

To show $\mathbb{R}_{coh} \models \varphi$, take any $S \in Sk\,Coh$ (!) such that $S \models \mathbb{R}_{coh}$, to show $S \overset{?}{\models} \varphi$. We have that $S = sk_{coh}(\mathbb{C})$, for a (unique, <u>small</u>) coherent category $\mathbb{C}$ — this is because of the Proposition on top of page 54.

I am going to use Remark (i) on p. $\boxed{54}$ twice. We have $\varphi : A \to B$; let

$$f : A \to S$$

and

$$M : S \to sk_{COH}(Set)$$

be arbitrary morphisms in Sk COH. I will abbreviate $sk_{COH}(Set)$ as Set. I have the composite morphism

$$M \circ f : A \to Set.$$

By Remark (i), '$Set \models \varphi$' implies that the arrow $(M \circ f)(\alpha)$ is an isomorphism. We have that, for the arrow $f(\alpha)$ in $\mathbb{C}$ (in S), it is the case that for all $M : S \to Set$ in Sk COH, $M(f(\alpha))$ is an isomorphism. By 1), p. $\boxed{63}$, $f(\alpha)$ is an isomorphism. By Remark (i) again, since this holds for all $f : A \to S$, we conclude that $S \models \varphi$ as desired.

Now we know that $\mathbb{R}_{coh} \models \varphi$.

By the GCT, mentioned as tool 2) on top of p 64, this means that $\mathbb{R}_{coh} \vdash \varphi$

QED.

<u>Further remarks</u> (i) One is mainly interested in the case when $\varphi : A \to B$ in the theorem is finite, not just relatively finite. One calls a finite $\varphi : A \to B$ for which $\mathbb{R}_{coh} \models \varphi$ (equivalently, $\mathbb{R}_{coh} \vdash \varphi$) a coherent validity; the class of coherent validities is Valid_{coh}. The class of <u>first-order</u> coherent validities is the subset $\mathrm{Valid}_{coh}^{f.o.}$.

Both of these classes of arrows are closed under isomorphism: if $A \xrightarrow{\varphi} B$ is in the class, and

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \cong\downarrow & \circleddash & \downarrow\cong \\ A' & \xrightarrow[\varphi']{} & B' \end{array}$$

, then $A' \xrightarrow{\varphi'} B'$ is in the class.

class as well.

Now, recall the set $\mathbb{HF}$ of hereditarily finite (hf) sets. We have set up our notions so that all specifications templates $\overline{K}_{id}, \dots$, and all coherent sketch-ations $Ax_0, \dots, Ax_{27}$ are elements of $\mathbb{HF}$ — under the usual set-theoretic construction of notions such as ordered pairs, functions, etc. (for instance, a graph is a tuple (Ob, Arr, dom, codom) where Ob, Arr are sets, dom and codom are functions Ob $\to$ Arr; by a tuple I mean a sequence (string), i.e., a function with domain some $n = \{0, \dots, n-1\}$). Moreover, pushouts of diagrams of (finite) sets and arrows between them can be effectively (recursively) constructed as further finite sets: if we start with hf sets and functions, the result is also hf, in particular. It is obvious that all finite sets in the world are isomorphic to ones which are hf — and the category of hf sets and hf functions between them is a full subcategory of the category of all (finite) sets. We see that every <u>deduction</u> of an hf entailment φ from $\mathbb{R}_{coh}$ is <u>isomorphic</u> to one that itself (construed as a sequence of sequences) is an hf set. One then sees that the concept

"$d \in HF$ is a deduction of $\varphi \in HF$ from $\mathbb{R}_{coh}$"

is a <u>recursive</u> relation $Ded(\varphi, d)$ on HF

It follows that

$$Valid_{coh} \cap HF$$

and

$$Valid^{f.v.}_{coh} \cap HF$$

are recursively enumerable sets.

(ii) The completeness theorem of p. 61 is formulated for all relatively small first order entailments φ. I leave it to the reader to see that, in this generality, it implies the coherent completeness we used in the proof of it.