Makkai's talk on

December 9th, 2014

[MP]: M. Makkai & A. M. Pitts,
"Some results on locally finitely presentable categories",
Trans A. M. S. 299, no. 2,
pp 473 – 496 (1987).

$\boxed{\text{Proposition}}$ ([MP]-1987) (essentially)

Let $\mathbb{C}$, $\mathbb{D}$ be small categories with finite limits

$$\underline{I} : \mathbb{C} \longrightarrow \mathbb{D}$$

a lex functor (preserving finite limits)

Suppose that

$$\underline{I}^* = (-)\circ \underline{I} : LEX(\mathbb{D}, Set) \longrightarrow LEX(\mathbb{C}, Set)$$

is full and faithful.

Then $\underline{I}$ has the left lifting property with respect to all inverse-image functors of surjective geometric morphisms between Grothendieck toposes:

For any commutative square of solid arrows

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, "\underline{I}"'] \arrow[dr, phantom, "\cong", near start] & \mathcal{E} \arrow[d, "\pi^*"] \\
\mathbb{D} \arrow[r, "G"'] \arrow[ur, dashed, "H"] \arrow[ur, phantom, "\cong", near end, shift right=3] & \widetilde{\mathcal{F}}
\end{tikzcd}$$

in which F$\mathcal{E}$, and $\mathcal{F}$ Grothendieck toposes, π^* is the inverse-image functor of a surjective geometric morphisms — i.e., π^* is conservative: $\pi^*(f)$ is an isomorphism implies f is an isomorphism, for all arrows f in $\mathcal{E}$ —, and F and G are lex functors, there is a diagonal H, a lex functor, making the two triangles commute up to isomorphism

Proof: Consider:

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "y_{\mathbb{C}}"] \arrow[d, "I"'] \arrow[rr, bend left=40, "F"] \arrow[dr, phantom, "\cong"] & \hat{\mathbb{C}} \arrow[r, "f^*"] \arrow[d, "i^*"] \arrow[dr, phantom, "\cong"] \arrow[r, phantom, bend left=30, "\cong"] & \mathcal{E} \arrow[d, "\pi^*"] \\
\mathbb{D} \arrow[r, "y_{\mathbb{D}}"'] \arrow[rr, bend right=40, "G"'] & \hat{\mathbb{D}} \arrow[r, "g^*"'] \arrow[r, phantom, bend right=30, "\cong"] & \mathcal{F}
\end{tikzcd}
$$

Here: $\hat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{op}}$: presheaves

$\hat{\mathbb{D}}$ similarly

$y_{\mathbb{C}}, y_{\mathbb{C}}$: Yoneda functors

$$i^* = \mathrm{Lan}_{y_{\mathbb{C}}}(y_{\mathbb{D}} I) =$$

= the left Kan extension of

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "y_{\mathbb{C}}"] \arrow[dr, "y_{\mathbb{D}} I"'] & \hat{\mathbb{C}} \arrow[d, "i^*"] \\
& \hat{\mathbb{D}}
\end{tikzcd}$$

We have that the left-hand square iso-commutes (i.e., the last triangle) (commutes up to isomorphism)

Similarly,

$$f^* = \mathrm{Lan}_{y_{\mathbb{C}}}(\mathbb{F})$$

$$g^* = \mathrm{Lan}_{y_{\mathbb{D}}}(\mathbb{G})$$

and the upper & lower (curved) triangles iso-commute.

As left Kan extensions of lex functors,

i^*, f^*, g^* are left exact. They also

inverse-image functors of geometric morphisms, since they are cocontinuous (preserve small colimits). In particular, i, the right adjoint of i^*, $i^* \dashv i$, is

$$i = (-)\circ I^{op} : \hat{\mathbb{D}} \longrightarrow \hat{\mathbb{C}}$$

$$\mathbb{D}^{op} \xrightarrow{\Phi} Set \longmapsto \mathbb{C}^{op} \xrightarrow{I^{op}} \mathbb{D}^{op} \xrightarrow{\Phi} Set$$

(up to isomorphism)

The right-hand square iso-commutes: the assumption (the outside quadrangle iso-commutes) implies that the two functors $\hat{\mathbb{C}} \overset{\pi^* f^*}{\underset{g^* i^*}{\rightrightarrows}} \mathcal{F}$ precomposed with $y_{\mathbb{C}}$ are isomorphic. Since these two functors are cocontinuous, and $\mathbb{C} \xrightarrow[y_{\mathbb{C}}]{} \hat{\mathbb{C}}$ is the free completion of $\mathbb{C}$ to a category having small colimits, it follows that $\pi^* f^* \cong g^* i^*$.

<u>Lemma</u> Assuming that $I^* = (-)\circ I : LEX(\mathbb{D}, Set) \to LEX(\mathbb{C}, Set)$ we have that $i = (-)\circ I^{op} : \hat{\mathbb{D}} \longrightarrow \hat{\mathbb{C}}$

is full and faithful

(For the proof of the lemma, see below)

In topos terminology, the conclusion of the lemma is that the geometric morphism $(i_*, i^*) : \hat{\mathbb{D}} \longrightarrow \hat{\mathbb{C}}$ is an inclusion. A rather simple argument shows the (well known) fact that $i^* : \hat{\mathbb{C}} \to \hat{\mathbb{D}}$ is a quotient morphism in the (2-)category of Grothendieck toposes:

for any Grothendieck topos $\mathcal{G}$ and inverse-image functor $\hat{\mathbb{C}} \xrightarrow{\ell^*} \mathcal{G}$, such that, for all arrows f in $\hat{\mathbb{C}}$,

$$i^*(f) \text{ is an iso} \Rightarrow \ell^*(f) \text{ is an iso}$$

(Write $i^* \overset{iso}{\Rightarrow} \ell^*$)

there is inverse-image functor $\hat{\mathbb{D}} \xrightarrow{m^*} \mathcal{G}$, such that

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[dr, "\ell^*"] \arrow[d, "i^*"'] & \\ \hat{\mathbb{D}} \arrow[r, "m^*"'] \arrow[ur, phantom, "\cong"] & \mathcal{G} \end{tikzcd}$$

and m^* is unique up to isomorphism

Using that π^* is conservative ($\pi^* \overset{iso}{\Rightarrow} Id_{\mathcal{E}}$), and that, of course, $i^* \overset{iso}{\Rightarrow} g^* i^*$, we have, from the commutativity of right-hand square, that $i^* \overset{iso}{\Rightarrow} f^*$; hence there is h^*

$$
\begin{tikzcd}
\hat{\mathbb{C}} \arrow[r, "f^*"] \arrow[d, "i^*"'] \arrow[dr, phantom, "\cong" near start] & \mathcal{E} \\
\hat{\mathbb{D}} \arrow[ur, "h^*"'] &
\end{tikzcd}
$$

Then the two functors

$$
\begin{tikzcd}
\hat{\mathbb{D}} \arrow[r, shift left, "\pi^* h^*"] \arrow[r, shift right, "g^*"'] & \mathcal{F}
\end{tikzcd}
$$

when precomposed with i^*, become isomorphic; hence, by the uniqueness part in the notion of "quotient", the lower triangle in

$$
\begin{tikzcd}
\hat{\mathbb{C}} \arrow[r, "f^*"] \arrow[d, "i^*"'] \arrow[dr, phantom, "\cong" near start] & \mathcal{E} \arrow[d] \\
\hat{\mathbb{D}} \arrow[ur, "h^*"] \arrow[r] \arrow[ur, phantom, "\cong" near end] & \mathcal{F}
\end{tikzcd}
$$

also iso-commutes. Looking at our diagrams, $H \underset{def}{=} h^* y_{\mathbb{D}}$

answers the demand of the Proposition.

<u>On the proof of the lemma</u>:

~~Sub~~ Suppose $I^*: \underbrace{LEX(\mathbb{D}, Set)}_{\mathbb{D}^\#} \longrightarrow \underbrace{LEX(\mathbb{C}, Set)}_{\mathbb{C}^\#}$

is full & faithful.

First we show that for all $C \in \mathbb{C}$, and for $I^C: \mathbb{C} \downarrow C \longrightarrow \mathbb{D} \downarrow I(C)$

induced by I, a lex functor between lex categories,

1⊙ $(I^C)^*: LEX(\mathbb{D} \downarrow I(C), Set) \longrightarrow LEX(\mathbb{C} \downarrow C, Set)$

is full and faithful (I omit the proof).

2⊙ Next, I show that, under the standing hypothesis, every object D of $\mathbb{D}$ is a retract of $I(C)$ for some C in $\mathbb{C}$: there is

$$D \underset{j}{\overset{q}{\rightleftarrows}} I(C) \quad , \quad qj = 1_D$$

as follows: consider the left adjoint $I_!: \mathbb{C}^\# \longrightarrow \mathbb{D}^\#$ of I^*. ~~For~~ $C \in \mathbb{C}$, $\hat{C} \in \mathbb{C}^\#$, and $\underbrace{I_!(\hat{C})}\cong\underbrace{\widehat{I(C)}}$

$$\left.\begin{array}{r}\mathbb{D}(-,I(C))\\ \hat{C}=\mathbb{C}(-,C)\end{array}\right\}\text{ (reminder)}$$

Given D, consider $\hat{D}=\mathbb{D}(-,D)\in\mathbb{D}^{\#}$, and $I^*(\hat{D})\in\mathbb{C}^{\#}$. In $\mathbb{C}^{\#}$, every object is a filtered colimit of the representables,

$$I^*(\hat{D})\cong\operatorname*{colim}_{i\in I}\hat{C}_i$$

Hence

$$I_!\,I^*(\hat{D})\cong I_!\operatorname*{colim}_{i\in I}\hat{C}_i$$

$$\cong\operatorname*{colim}_{i\in I}I_!\,\hat{C}_i$$

But, since I^* is full and faithful,

$$I_!\,I^*(\hat{D})\xrightarrow[\varepsilon_{\hat{D}}]{\cong}\hat{D}$$

is an isomorphism ($\mathbb{D}^{\#}$ is a reflective subcategory of $\mathbb{C}^{\#}$)

Thus

$$\hat{D}\cong\operatorname*{colim}_{i\in I}I_!\,\hat{C}_i\cong\operatorname*{colim}_{i\in I}\widehat{I(C_i)}$$

But $\hat{D}$ is finitely presentable (we are in $\mathbb{D}^\#$), and the colimit is a filtered one:

$$\operatorname*{colim}_{i\in I} \mathbb{D}(\hat{D}, \widehat{I(C_i)}) \xrightarrow{\cong} \mathbb{D}(\hat{D}, \operatorname*{colim}_{i\in I} \widehat{I(C_i)})$$

is an <u>isomorphism</u>; in particular, surjective

Thus the isomorphism $\hat{D} \xrightarrow[\varphi]{\cong} \operatorname*{colim}_i \widehat{I(C_i)}$

factors, for some $i \in I$, as

$$\begin{tikzcd} \hat{D} \arrow[rr, "\cong", "\varphi"'] \arrow[dr, "\hat{J}"'] \arrow[rr, phantom, shift right=4ex, "="] & & \operatorname*{colim}_i \widehat{I(C_i)} \\ & \widehat{I(C_i)} \arrow[ur, "\gamma_i"'] & \end{tikzcd}$$

(γ_i: colimit coprojection)

and
$$\begin{tikzcd} \hat{D} \arrow[r, shift right, "\hat{J}"'] & \widehat{I(C_i)} \arrow[l, shift right, "\varphi^{-1}\gamma_i = \hat{q}"'] \end{tikzcd}$$

shows that $\hat{D}$ is a retract of $\widehat{I(C_i)}$.

By Yoneda, D is a retract of $I(C_i)$. <u>2⊙: done</u>

But, we can apply 2⊙ to I^c instead of I since by 1⊙, $(I^c)^*$ is full and faithful

We conclude:

3① I^* full & faithful implies:

for all $C \in \mathbb{C}$, every object of $\mathbb{D} \downarrow I(C)$ is a retract of an object of the form $I^C(A)$, A an object of $\mathbb{C} \downarrow C$.

In other words, given $C \in \mathbb{C}$, and

$$D \xrightarrow{f} I(C) \quad \text{in } \mathbb{D}$$

there are

$$C' \xrightarrow{g} C \quad \text{in } \mathbb{C}$$

and

$$\begin{tikzcd} D \arrow[dr, "f"] \arrow[d, shift right, "j"'] & \\ I(C') \arrow[u, shift right, "p"'] \arrow[r, "I(g)"'] & I(C) \end{tikzcd}$$

such that $jp = 1_D$, $I(g)j = f$, $fp = I(g)$

4⊙ Next, there is an argument, involving a big diagram showing that the condition in 3⊙ implies, also using that $I : \mathbb{C} \to \mathbb{D}$ is a finite product preserving functor between categories with finite products, that

for any category $\mathbb{X}$,

$$\mathbb{X}^{\mathbb{D}} = [\mathbb{D}, \mathbb{X}] \xrightarrow[(-)\circ I]{} [\mathbb{C}, \mathbb{X}]$$

is full and faithful. Apply this to $\mathbb{X} = \mathrm{Set}^{op}$, and obtain the assertion of the lemma. — and that of the $\boxed{\text{Proposition}}$ (p. 1).

NB: There is a more direct proof of the Proposition, using "conceptual completeness", in the precise sense mentioned in [MP] (on p. 188, two paragraph up from 2.6 Proposition) – his proof (mentioned in the talk) is <u>not</u> mentioned in [MP]

Th 3.3 [MP]

$\mathbb{C}$: small cat with finite limits

$\mathcal{E}$: Gr. topos with enough pts

$\mathbb{C} \xrightarrow{F} \mathcal{E}$: lex

$\mathcal{E} \xrightarrow{L} Set$: lex

$$LF \in \underbrace{\left[\{ F_p \underset{def}{=} p^* F : p \in Pt(\mathcal{E}) \} \right]}_{\text{call this } \underset{\sim}{K}} \subset LEX(\mathbb{C}, Set)$$

lim/filt colim

need definition!

<u>Def.</u> A point of $\mathcal{E}$ is $p = (p_*, p^*) : Set \longrightarrow \mathcal{E}$

geom morph ; i.e., just

$$p^* : \mathcal{E} \longrightarrow Set$$

cocts & lex

<u>Proof</u> (Uses 2.4!)

Construct diagram:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d, "I"'] \arrow[dr, phantom, "\cong"] & \mathcal{E} \arrow[d, "\pi^*"] \\ \mathbb{D} \arrow[r, "G"'] & \mathcal{F} \end{tikzcd}$$

as follows. The category $\underset{\sim}{K} = [\, F_p : p \in Pt(\mathcal{E}) \,]$ is a full subcat of $\mathbb{C}^\# = LEX(\mathbb{C}, Set)$, closed under limits and filtered colimits. By [MP] 2.4, $\underset{\sim}{K}$ is a lfp. category; $\underset{\sim}{K} \simeq LEX(\mathbb{D}, Set) = \mathbb{D}^\#$ a small lex cat $\mathbb{D}$.

Write
$$\begin{tikzcd} \mathbb{D}^\# \arrow[r, "\simeq", "\varepsilon"'] \arrow[rr, bend right, "\Phi = \mathrm{incl} \circ \varepsilon"'] & \underset{\sim}{K} \arrow[r, hook] & \mathbb{C}^\# \end{tikzcd}$$

$\Phi : \mathbb{D}^\# \longrightarrow \mathbb{C}^\#$ is a limit / filtered colimit preserving functor;

by basic duality theory, there is

$$\mathbb{C} \xrightarrow{I} \mathbb{D} \quad \text{lex}$$

such that $I^*: \mathbb{D}^\# \longrightarrow \mathbb{C}^\#$ is isomorphic to Φ:

$$\boxed{\begin{tikzcd} \mathbb{D}^\# \arrow[r, bend left, "I^*"] \arrow[r, bend right, "\Phi"'] \arrow[r, phantom, "\downarrow\cong"] & \mathbb{C}^\# \end{tikzcd}}$$

we've defined $\mathbb{D}$ and I

Now: let P be a set of points of $\mathcal{E}$

such that the set

$$\{\mathcal{E} \xrightarrow[p^*]{} Set : p \in P\}$$

is jointly conservative: if for an arrow f in $\mathcal{E}$

$p^*(f)$ is an iso for all $p \in P$, then f is an iso

(with $P = Pt(\mathcal{E})$, a proper class, this being true

is the definition of $\mathcal{E}$ having enough pts;

all coherent & separable toposes have enough pts)

(P can always be taken to be a set, in that case)

Let $\mathcal{F} = \mathrm{Set}^P$ (simple power: P: discrete cat)

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi^*"] \\ \mathcal{F} \end{tikzcd}$$

is defined in the obvious way

$$\begin{tikzcd} \mathcal{E} \arrow[d] \arrow[dr, "p^*"] & \\ \mathrm{Set}^P \arrow[r, "\pi_p"'] \arrow[ur, phantom, "\ominus"] & \mathrm{Set} \end{tikzcd}$$

π^* is conservative, by definition.

Now, for G:

let $p \in P$:

$F_p = p^* F \in \underset{\sim}{K}$

Let $G_p \in \mathbb{D}^{\#}$ s.t. $\varepsilon G_p \cong F_p$ (ε is essentially surjective)

Then: $\underset{\smile}{G_p I} = I^*(G_p) \cong \Phi(G_p) = \varepsilon G_p \cong \underset{\smile}{p^* F}$

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, "I"'] \arrow[dr, phantom, "\therefore \cong"] & \mathcal{E} \arrow[d, "\pi^*"] \arrow[dr, bend left, "p^*"] \arrow[dr, phantom, "\cong"] & \\
\mathbb{D} \arrow[r, "G"] \arrow[rr, bend right, "G_P"'] & \mathbf{Set}^P \arrow[r, "\pi_P"] \arrow[d, phantom, "="] & \mathbf{Set} \\
 & &
\end{tikzcd}
$$

This defines G, and show $\cong$.

By the $\boxed{\text{Proposition}}$:

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, "I"'] \arrow[dr, phantom, "\cong"] & \mathcal{E} \arrow[r, "L"] & \mathbf{Set} \\
\mathbb{D} \arrow[ur, "H"'] & &
\end{tikzcd}
$$

Thus:

$$LF \cong LHI = I^*(LH)$$

$LH \in \mathbb{D}^\#$

$$\cong \Phi(LH) = \varepsilon(LH)$$

$$\in \underset{\sim}{K}$$

by definition!

Corollary 3.4 [MP]

$\mathcal{C} = (\mathbb{C}, J)$

subcanonical topology; i.e., for each

$C \in \mathbb{C}$, $\hat{C}$ is a J-sheaf. Assume $\Rightarrow$ assertion: next page!

the presheaf

[NB] $\mathcal{C}$ has enough models!, i.e., $\tilde{\mathcal{C}}$ has enough points.

[NB: $\mathrm{Mod}_{\mathrm{Set}}(\mathcal{C})$: the category of all $M: \mathbb{C} \to \mathrm{Set}$

s.t.

1) M is left exact

2) M takes every J-cover into a

surjective family in

$S \in J_U \quad S \subseteq \hat{U}$

$\{M(V) \xrightarrow[M(f)]{} M(U) : V \xrightarrow{f} U \in S\}$

is surjective!

'$\mathcal{C}$ has enough models' means:

for $\mathcal{E} = \tilde{\mathcal{C}} = \mathrm{Sh}(\mathcal{C})$, we saw this before!

for instance: separable: always

we have: $\mathrm{Mod}_{Set}(\mathcal{E}) \subseteq \mathrm{Lex}(\mathbb{C}, Set)$

assertion: $[\ -''-\ ]_{lim/filt\,colim} \subseteq \mathrm{Lex}(\mathbb{C}, Set)$

proof:

$$
\begin{tikzcd}
\mathbb{C} \arrow[rr, "y_{\mathbb{C}}"] \arrow[dr, "H"'] \arrow[ddddr, bend right, "\hat{C}"'] & & \hat{\mathbb{C}} \\
& \tilde{\mathcal{E}} \arrow[ur, hook, "i"'] \arrow[d, "\Delta_C"] & \\
& \tilde{\mathcal{E}}/H(C) \arrow[dd] & \\
& & \\
& \Gamma &
\end{tikzcd}
$$

$\cong$

this because of subcanonical

right adjoint to

$$
\begin{tikzcd}
\tilde{\mathcal{E}} \\
\tilde{\mathcal{E}}/H(C) \arrow[u, "\Sigma\ (\text{forget})"']
\end{tikzcd}
$$

[what is Δ_C?]

$A \in \mathbb{C}$: because: $\Gamma \Delta_C H(A) \cong (\tilde{\mathcal{E}} \downarrow H(C))(1, \Delta_C H(A))$

because $\Gamma(X) = (\tilde{\mathcal{E}} \downarrow H(C))(1, X)$

$\cong \tilde{\mathcal{E}}(\Sigma(1), H(A)) = \tilde{\mathcal{E}}(H(C), H(A))$

$\cong \hat{\mathbb{C}}(H(C), H(A)) \cong \mathbb{C}(C, A)$

(what is Δ_C?)

Now, apply

3.3

with

$$\mathcal{E} = \tilde{\mathcal{C}}$$

$$F = H$$

$$\text{and } L = \Gamma \Delta_U$$

What do you get?

get:

$$\underline{\underline{\hat{C}}} \simeq \Gamma\Delta_C \circ H \in \underline{\underline{[H_p \mid p \in Pt(\mathcal{E})]}}$$

by def. of $\mathcal{E}$,

$$\{H_p : p \in Pt(\tilde{\mathcal{C}})\} = Ob(Mod_{Set}(\mathcal{C}))$$

why? because

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[dr, "M"'] & \tilde{\mathcal{C}} \arrow[d, "p^*"] \arrow[dl, phantom, "\cong"] \\ & Set \end{tikzcd} \qquad H_p = p^* H$$

We have proved: all representables are

in limit / filtered colimit closure.

Of course, all objects of $Lex(\mathbb{C}, \mathbf{Set})$

are filtered colimits of representables

This proves the equality on top of page 13.