

Makkai's talk on

December 9th, 2014

[MP]: M. Makkai & A. M. Pitts

"Some results on locally
finitely presentable categories",

Trans A. M. S. 299, no. 2,

pp 473-496 (1987).

Proposition (MP-1987) (essentially)

1

Let $\mathcal{C}, \mathcal{D}$ be small categories
with finite limits

$$I: \mathcal{C} \longrightarrow \mathcal{D}$$

a lex functor (preserving finite limits)

Suppose that

$$I^* = (-) \circ I: \text{LEX}(\mathcal{D}, \text{Set}) \longrightarrow \text{LEX}(\mathcal{C}, \text{Set})$$

is full and faithful.

Then I has the left lifting property
with respect to all inverse-image functors
of subjective geometric morphisms
between Grothendieck toposes:

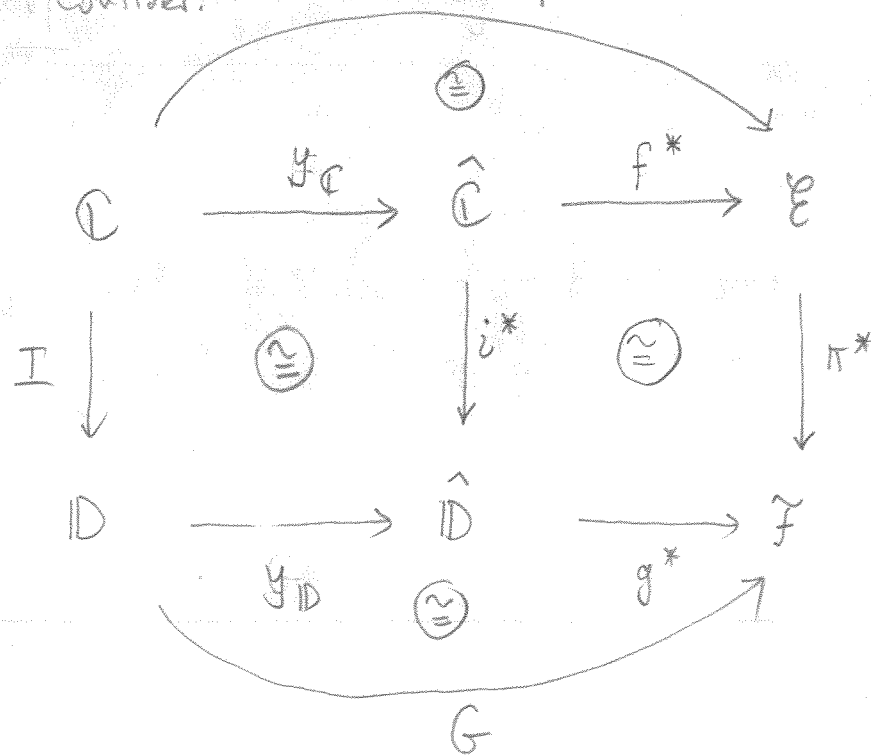
For any commutative square of solid
arrows

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ I \downarrow & \nearrow \text{---} \gamma & \downarrow \pi^* \\ \mathcal{D} & \xrightarrow{G} & \tilde{\mathcal{F}} \end{array}$$

$\begin{pmatrix} \cong \\ \cong \end{pmatrix}$
 $\begin{pmatrix} \cong \\ \cong \end{pmatrix}$

in which $\mathcal{F}\mathcal{E}$, and $\mathcal{F}$ Grothendieck
toposes, π^* is the inverse-image functor
of a surjective geometric morphism — i.e.,
 π^* is conservative: $\pi^*(f)$ is an isomorphism
implies f is an isomorphism, for all arrows f in $\mathcal{E}$ —,
and F and G are lex functors, there is
a diagonal H , a lex functor, making the
two triangles commute up to isomorphism

TP_{10.2} Consider: $\mathcal{E} \xrightarrow{f^*} \mathcal{F}$



Here: $\hat{\mathbb{C}} = \text{Set}^{\mathbb{C}^{\text{op}}}$: presheaves

$\hat{\mathbb{D}}$ similarly

$y_{\mathbb{C}}, y_{\mathbb{D}}$: Yoneda functors

$$i^* = \text{Lan}_{y_{\mathbb{C}}} (y_{\mathbb{D}} I) =$$

= the left Kan extension of

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{y_{\mathbb{C}}} & \hat{\mathbb{C}} \\ & \searrow y_{\mathbb{D}} I & \downarrow i^* \\ & & \hat{\mathbb{D}} \end{array}$$

We have that the left-hand square
150-commutes

(i.e., the triangle) (commutes up to isomorphism)
last

Similarly,

$$f^* = \text{Lan}_{y_{\mathbb{C}}} (F)$$

$$g^* = \text{Lan}_{y_{\mathbb{D}}} (G)$$

and the upper & lower (curved) triangles iso-commute.

As left Kan extensions of lex functors,

i^*, f^*, g^* are left exact. They also

inverse image functors of geometric morphisms,

Since they are ω -continuous (preserve small colimits). In particular, i , the right adjoint of i^* , $i^* + i$, is

$$i = (-) \circ I^{op} : \hat{\mathbb{D}} \longrightarrow \hat{\mathbb{C}}$$

$$\mathbb{D}^{op} \xrightarrow{\Phi} \mathbf{Set} \quad \xrightarrow{\quad} \quad \mathbb{C}^{op} \xrightarrow{I^{op}} \mathbb{D}^{op} \xrightarrow{\Phi} \mathbf{Set}$$

(up to isomorphism)

The right-hand square is commutative: the assumption (the outside quadrangle is commutative) implies that

the two functors $\hat{\mathbb{C}} \xrightarrow[\pi^* f^*]{g^* i^*} \mathbf{F}$ precomposed

with $y_{\mathbb{C}}$ are isomorphic. Since these two functors are ω -continuous, and $\mathbb{C} \xrightarrow{y_{\mathbb{C}}} \hat{\mathbb{C}}$ is the

free completion of $\mathbb{C}$ to a category having small colimits, it follows that $\pi^* f^* \cong g^* i^*$.

Lemma Assuming that $I^* = (-) \circ I : \mathbf{LEX}(\mathbb{D}, \mathbf{Set}) \rightarrow \mathbf{LEX}(\mathbb{C}, \mathbf{Set})$

we have that $i = (-) \circ I^{op} : \hat{\mathbb{D}} \xrightarrow{1+} \hat{\mathbb{C}}$

is full and faithful

(For the proof of the lemma, see below)

In topos terminology, the lemma is that

conclusion of the

the geometric morphism $(i, i^*) : \hat{\mathbb{D}} \rightarrow \hat{\mathbb{C}}$

is an inclusion. A rather simple argument

shows the (well known) fact that $i^* : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{D}}$

is a quotient morphism in the 2-category of

Grothendieck toposes:

for any Grothendieck topos $\mathcal{G}$ and
inverse-image functor $\hat{\mathbb{C}} \xrightarrow{e^*} \mathcal{G}$, there is inverse-image

functor $\hat{\mathbb{D}} \xrightarrow{m^*} \mathcal{G}$, such that

$$\begin{array}{ccc} \hat{\mathbb{C}} & & \\ i^* \downarrow & \searrow e^* & \\ \hat{\mathbb{D}} & \xrightarrow{m^*} & \mathcal{G} \end{array} \quad (\cong)$$

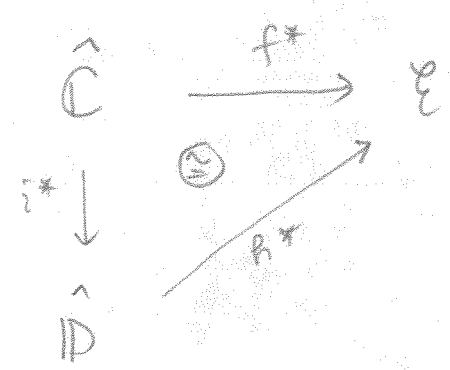
such that, for all arrows
 f in $\hat{\mathbb{C}}$, i

$i^*(f)$ is an iso $\Rightarrow e^*(f)$ is an iso

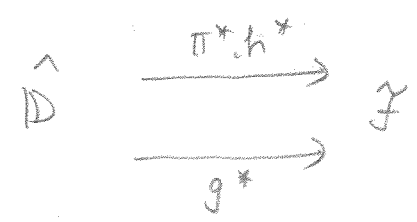
(write $(i^*)_+ \xrightarrow{\text{iso}} e^*$)

and m^* is unique up to isomorphism

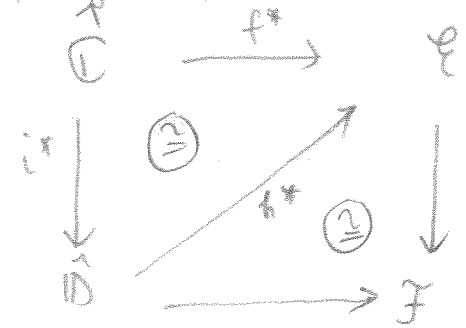
Using that π^* is conservative ($\pi^* \xRightarrow{\text{iso}} \text{Id}_{\mathcal{E}}$),
 and that, of course, $i^* \xRightarrow{\text{iso}} g^* i^*$, we have,
 from the commutativity of right-hand square,
 that $i^* \xRightarrow{\text{iso}} f^*$; hence there is h^*



Then the two functors



when precomposed with i^* , become isomorphic;
 hence, by the uniqueness part in the notion of "quotient",
 the lower triangle in



also iso-commutes. Looking at our diagrams, $H_{\text{def}} = h^* y_{\mathbb{D}}$

answer the demand of the Proposition.

On the proof of the lemma:

Sub (Suppose $I^*: \underbrace{\text{LEX}(\mathbb{D}, \text{Set})}_{\mathbb{D}^\#} \rightarrow \underbrace{\text{LEX}(\mathbb{C}, \text{Set})}_{\mathbb{C}^\#}$)
is full & faithful.

First we show that for all $C \in \mathbb{C}$, and

$$\text{for } I^C: \mathbb{C} \downarrow C \longrightarrow \mathbb{D} \downarrow I(C)$$

induced by I , a lex functor between lex categories,

$$1 \textcircled{\circ} \quad (I^C)^*: \text{LEX}(\mathbb{D} \downarrow I(C), \text{Set}) \longrightarrow \text{LEX}(\mathbb{C} \downarrow C, \text{Set})$$

is full and faithful (I omit the proof).

2 \textcircled{\circ} Next, I show that, under the standing hypothesis, every object D of $\mathbb{D}$ is a retract of $I(C)$ for some C in $\mathbb{C}$: there is

$$D \begin{array}{c} \xleftarrow{q} \\ \xrightarrow{j} \end{array} I(C), \quad qj = 1_D$$

as follows: consider the left adjoint $I_!: \mathbb{C}^\# \rightarrow \mathbb{D}^\#$

of I^* . For $C \in \mathbb{C}$, $\hat{C} \in \mathbb{C}^\#$,

$$\text{and } \underbrace{I_! (\hat{C})} \cong \underbrace{I(\hat{C})}$$

$$\left. \begin{array}{l} ID(-, I(C)) \\ \hat{C} = C(-, C) \end{array} \right\} \text{ (reminders)}$$

Given D , consider $\hat{D} = ID(-, D) \in ID^\#$,

and $I^*(\hat{D}) \in C^\#$. In $C^\#$, every

object is a filtered colimit of the representables,

$$I^*(\hat{D}) \cong \operatorname{colim}_{i \in I} \hat{C}_i$$

Hence

$$I_! I^*(\hat{D}) \cong I_! \operatorname{colim}_{i \in I} \hat{C}_i$$

$$\cong \operatorname{colim}_{i \in I} I_! \hat{C}_i$$

But, since I^* is full and faithful,

$$I_! I^*(\hat{D}) \xrightarrow[\varepsilon_{\hat{D}}]{\cong} \hat{D}$$

is an isomorphism ($ID^\#$ is a reflective subcategory of $C^\#$)

$$\text{Thus } \hat{D} \cong \operatorname{colim}_{i \in I} I_! \hat{C}_i \cong \operatorname{colim}_{i \in I} \hat{I(C_i)}$$

But $\hat{D}$ is finitely presentable (we are in $\mathcal{D}^\#$), and the colimit is a filtered one:

$$\operatorname{colim}_{i \in I} \mathcal{D}(\hat{D}, I(\hat{C}_i)) \xrightarrow{\cong} \mathcal{D}(\hat{D}, \operatorname{colim}_{i \in I} I(\hat{C}_i))$$

is an isomorphism, in particular, surjective

Thus the isomorphism $\hat{D} \xrightarrow[\varphi]{\cong} \operatorname{colim}_i I(\hat{C}_i)$

factors, for some $i \in I$, as

$$\begin{array}{ccc} \hat{D} & \xrightarrow[\varphi]{\cong} & \operatorname{colim}_i I(\hat{C}_i) \\ & \searrow \hat{j} \quad \nearrow \gamma_i & \\ & I(\hat{C}_i) & \end{array}$$

(γ_i : colimit co-projection)
and

$$\begin{array}{ccc} \hat{D} & \xleftarrow[\hat{j}]{\varphi^{-1} \gamma_i = \hat{q}} & I(\hat{C}_i) \end{array}$$

shows that $\hat{D}$ is a retract of $I(\hat{C}_i)$.

By Yoneda, D is a retract of $I(C_i)$. 20: done

But, we can apply 20 to I^c instead of I
since by 10, $(I^c)^*$ is full and faithful

We conclude:

30 I^* full & faithful implies:

for all $C \in \mathcal{C}$, every object
of $\mathcal{D} \downarrow I(C)$ is a retract of an
object of the form $I^C(A)$, A an object
of $\mathcal{C} \downarrow C$.

In other words, given $C \in \mathcal{C}$, and

$$D \xrightarrow{f} I(C) \quad \text{in } \mathcal{D}$$

there are

$$C' \xrightarrow{g} C \quad \text{in } \mathcal{C}$$

and

$$\begin{array}{ccc} D & \xrightarrow{f} & I(C) \\ j \downarrow \uparrow p & & \\ I(C') & \xrightarrow{I(g)} & \end{array}$$

such that $j p = 1_D$, $I(g) j = f$, $f p = I(g)$

40 Next, there is an argument, involving a big diagram showing that the condition in 30 implies, also using that $I: \mathbb{C} \rightarrow \mathbb{D}$ is a finite product preserving functor between categories with finite products, that for any category $\mathbb{X}$,

$$\mathbb{X}^{\mathbb{D}} = [\mathbb{D}, \mathbb{X}] \xrightarrow{(-) \circ I} [\mathbb{C}, \mathbb{X}]$$

is full and faithful. Apply this to $\mathbb{X} = \text{Set}^{\text{op}}$, and obtain the assertion of the lemma. — and that of the Proposition (p. 11).

NB: There is a more direct proof of the Proposition, using "conceptual completeness", in the precise sense mentioned in [MP] (on p. 488, two paragraph up from 2.6 Proposition) — this proof (mentioned in the text) is not mentioned in [MP]

Th 3.3 [MP]



①: small cat with finite limit

$\mathcal{E}$: Gr. tops with enough pts

$$\textcircled{1} \xrightarrow{F} \mathcal{E} : \text{lex}$$

$$\mathcal{E} \xrightarrow{L} \text{Set} : \text{lex}$$

$$LF \in \left[\underbrace{\left\{ F_p = p^* F : p \in \text{Pt}(\mathcal{E}) \right\}}_{\text{need definition!}} \right] \subset \text{LEX}(\textcircled{1}, \text{Set})$$

lim/filt as lim

call this $K_{\sim}$

Def. A point of $\mathcal{E}$ is $p = (\tilde{p}, p^*) : \text{Set} \rightarrow \mathcal{E}$
geom morph; i.e., just

$$p^* : \mathcal{E} \rightarrow \text{Set}$$

cocts & lex

Proof (Uses 2.4!)

Commut diagram:

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\
 I \downarrow & \cong & \downarrow \pi^* \\
 \mathcal{D} & \xrightarrow{G} & \mathcal{F}
 \end{array}$$

as follows. The category $\mathcal{K} = [F_p : p \in \text{Pt}(\mathcal{E})]$ is a full subcat of $\mathcal{C}^\# = \text{LEX}(\mathcal{C}, \text{Set})$, closed under limits and filtered colimits. By [MP] 2.4, $\mathcal{K}$ is a lfp. category; $\mathcal{K} \cong \text{LEX}(\mathcal{D}, \text{Set}) = \mathcal{D}^\#$ a small lex cat $\mathcal{D}$.

$$\text{Write } \mathcal{D}^\# \xrightarrow[\varepsilon]{\cong} \mathcal{K} \hookrightarrow \mathcal{C}^\#$$

$$\Phi = \text{incl} \circ \varepsilon$$

$\Phi : \mathcal{D}^\# \longrightarrow \mathcal{C}^\#$ is a limit/filtered colimit preserving functor;

by basic duality theory, there is

$$\mathbb{Q} \xrightarrow{I} \mathbb{D} \text{ lex}$$

such that $I^*: \mathbb{D}^\# \rightarrow \mathbb{Q}^\#$ is isomorphic to $\mathcal{Q}$:

$$\begin{array}{ccc} \mathbb{D}^\# & \xrightarrow{I^*} & \mathbb{Q}^\# \\ \downarrow \cong & & \\ \mathcal{Q} & & \end{array}$$

we've defined $\mathbb{D}$ and I

Now: let P be a set of points of $\mathcal{E}$

such that the set

$$\{ \mathcal{E} \xrightarrow{p^*} \text{Set} : p \in P \}$$

for

is jointly conservative: if an arrow f in $\mathcal{E}$

$p^*(f)$ is an iso for all $p \in P$, then f is an iso

(with $P = \text{Pt}(\mathcal{E})$, a proper class; this being true

is the definition of $\mathcal{E}$ having enough pts;

all coherent & separable toposes have enough pts)

(P can always be taken to be a set, in that case)

11.4

Let $\mathcal{F} = \text{Set}^P$ (single power:
 P : discrete cat)

$\mathcal{E}$
 $\downarrow \pi^*$ is defined in the obvious way
 $\mathcal{F}$

$\mathcal{E} \xrightarrow{p^*} \text{Set}$
 $\downarrow \pi_p \circledast$

π^* is conservative, by
 definition.

Now, for G :

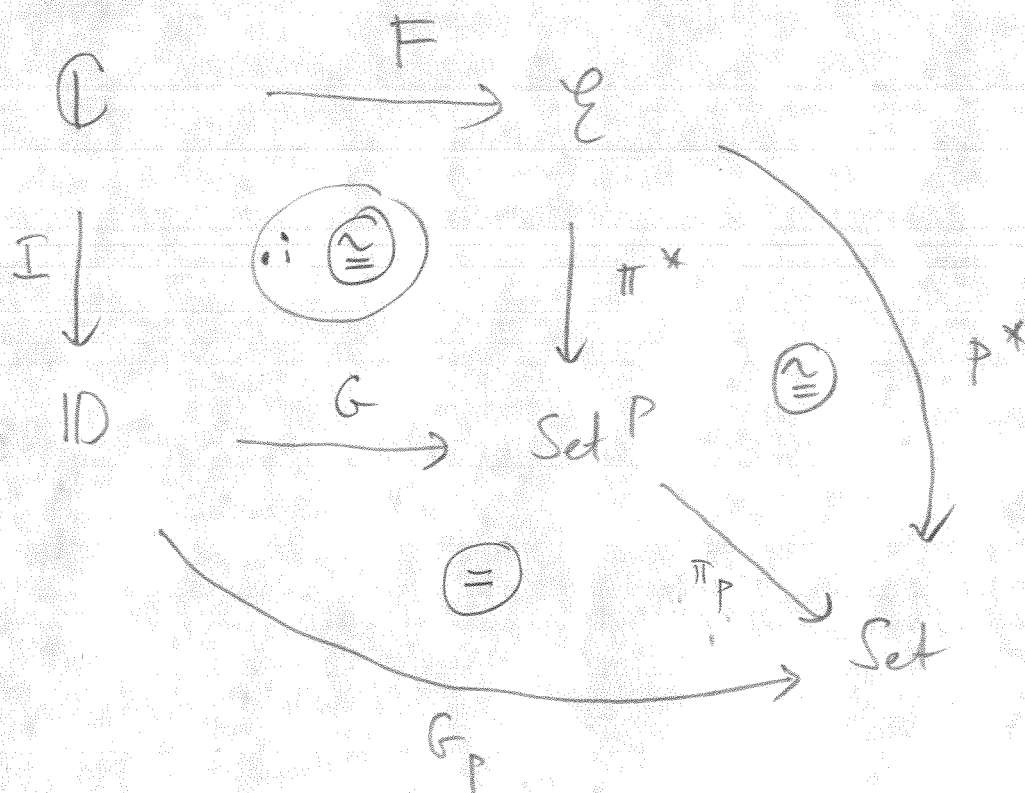
let $p \in P$:

$$F_p = p^* F \in \underset{\sim}{K}$$

Let $G_p \in \mathcal{D}^\#$ s.t. $\varepsilon G_p \cong F_p$ (ε is essentially
 surjective)

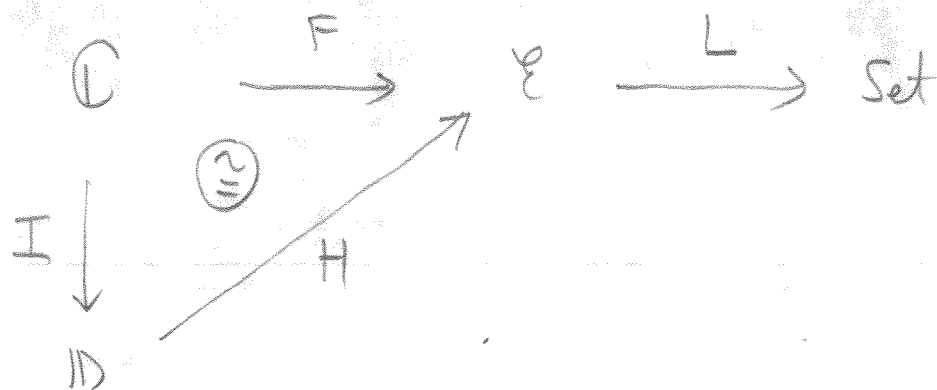
$$\text{Then: } \underbrace{G_p I} = I^*(G_p) \cong \Phi(G_p) = \varepsilon G_p \cong \underbrace{p^* F}$$

11.5



This defines G , and show $\cong$.

By the Proposition:



Thus:

$$LF \cong LH I = I^*(LH)$$

$$(LH \in \mathcal{D}^\#)$$

$$\cong \Phi(LH) = \varepsilon(LH)$$

$$\in \underset{\sim}{K}$$

by definition!

Corollary 3.4 [MP]

$$\mathcal{C} = (\mathcal{C}, J)$$

subcanonical topology; i.e., for each

$C \in \mathcal{C}$, $\hat{C}$ is a J -sheaf. Assume

(the presheaf)

assertion! next page!

[NB] $\mathcal{C}$ has enough models: i.e., $\hat{\mathcal{C}}$ has enough points.

[NB] $\text{Mod}_{\text{Set}}(\mathcal{C})$: the category of all $M: \mathcal{C} \rightarrow \text{Set}$

s.t.

1) M is left exact

2) M takes every J -cover into a

suspecting family in

$$S \in J_U \quad S \subseteq \hat{U}$$

$$\{M(V) \xrightarrow{M(f)} M(U) : V \overset{+}{\rightarrow} U \in S\}$$

is singleton:

' $\mathcal{C}$ has enough models' means:

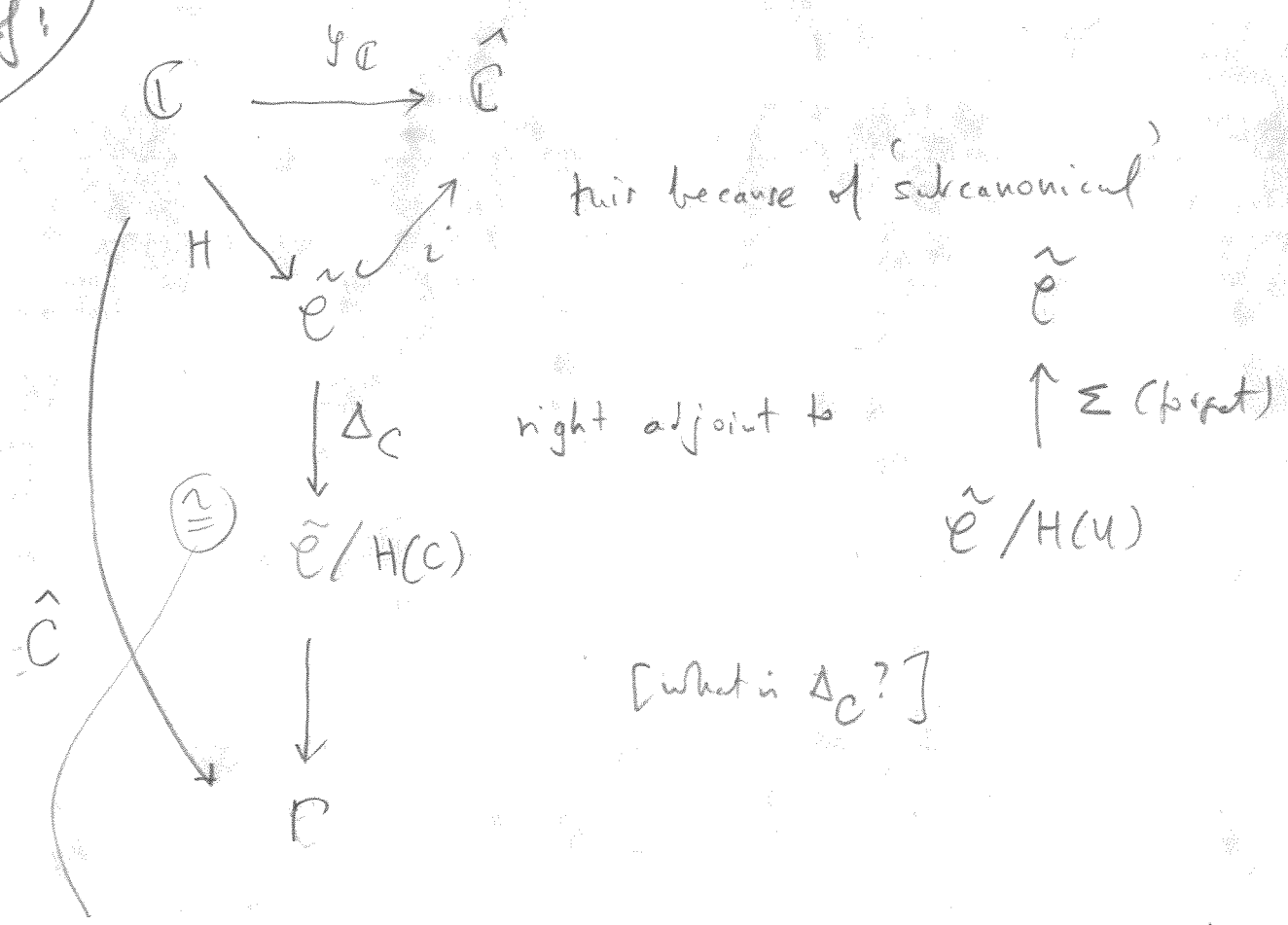
for $\mathcal{C} = \hat{\mathcal{C}} = \text{Sh}(\mathcal{C})$, we saw this before!

for instance: separable : always

we have: $\text{Mod}_{\text{Set}}(\mathcal{C}) \subseteq \text{Lex}(\mathcal{C}, \text{Set})$

Assertion: $\left[- \right]_{\text{lim/filt/olim}} \in \text{Lex}(\mathcal{C}, \text{Set})$

Proof:



$A \in \mathcal{C}$: because: $\Gamma_{\Delta_{\mathcal{C}}} H(A) \cong (\tilde{\mathcal{C}} \downarrow H(\mathcal{C}))(1, \Delta_{\mathcal{C}} H(A))$

because $\Gamma(X) = (\tilde{\mathcal{C}} \downarrow H(\mathcal{C}))(1, X)$

$\cong \tilde{\mathcal{C}}(\Sigma(1), H(A)) = \tilde{\mathcal{C}}(H(\mathcal{C}), H(A))$

$\cong \hat{\mathcal{C}}(H(\mathcal{C}), H(A)) \cong \mathcal{C}(\mathcal{C}, A)$

(what is Δ_C ?)

Now, apply

3.3

with

$$\mathcal{C} = \tilde{\mathcal{C}}$$

$$F = H$$

$$\text{and } L = \Gamma \Delta_U$$

What do you get?

get:

$$\underline{\hat{C}} \cong \Gamma \Delta_C \circ \underline{H \in [H_p \mid p \in \text{Pt}(\mathcal{C})]}$$

by def. of $\mathcal{C}$,

$$\{H_p : p \in \text{Pt}(\tilde{\mathcal{C}})\} = \text{ob}(\text{Mod}_{\text{Set}}(\mathcal{C}))$$

why? because

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \tilde{\mathcal{C}} \\ & \searrow \textcircled{\cong} & \downarrow p^* \\ M & & \text{Set} \end{array}$$

$$H_p = p^* H$$

We have proved: all representables are
in limit / filtered colimit closure.

Of course, all objects of $\text{Lex}(\mathcal{C}, \text{Set})$
are filtered colimits of representables

This proves the equality on top of page 17.