Strong conceptual completeness for Boolean coherent toposes

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What is strong conceptual completeness for first-order logic?

- A strong conceptual completeness statement for a logical doctrine is an assertion that a theory in this logical doctrine can be recovered from an appropriate structure formed by the models of the theory.
- Makkai proved such a theorem for first-order logic showing one could reconstruct a first-order theory T from $\mathbf{Mod}(T)$ equipped with structure induced by taking ultraproducts.
- Before we dive in, let's look at a well-known theorem from model theory, with the same flavor, which Makkai's result generalizes: the Beth definability theorem.

The Beth theorem

Theorem.

Let $L_0 \subseteq L_1$ be an inclusion of languages with no new sorts. Let T_1 be an L_1-theory. Let $F : \mathbf{Mod}(T_1) \to \mathbf{Mod}(\varnothing_{L_0})$ be the reduct functor. Suppose you know any of the following:

1. *There is a L_0-theory T_0 and a factorization:*

$$\begin{array}{ccc} \mathbf{Mod}(T_1) & \xrightarrow{F} & \mathbf{Mod}(\varnothing_{L_0}) \\ & \searrow^{\simeq} & \uparrow \\ & & \mathbf{Mod}(T_0) \end{array}$$

2. *F is full and faithful.*
3. *F is injective on objects.*
4. *F is full and faithful on automorphism groups.*
5. *F is full and faithful on $\mathrm{Hom}_{L_1}(M, M^{\mathcal{U}})$ for all $M \in \mathbf{Mod}(T_1)$ and all ultrafilters $\mathcal{U}$.*
6. *Every L_0-elementary map is an L_1-homomorphism of structures.*

<u>Then:</u> (*) Every L_1-formula is T_1-provably equivalent to an L_0-formula.

Useful consequence of Beth's theorem

Corollary.

Let T be an L-theory, let $\overline{S}$ be a finite product of sorts. Let $X : \mathbf{Mod}(T) \to \mathbf{Set}$ be a subfunctor of $M \mapsto \overline{S}(M)$.

Then: if X commutes with ultraproducts on the nose ("satisfies a Łos' theorem"), then X was definable, i.e. X is an evaluation functor for some definable set $\varphi \in \mathbf{Def}(T)$.

Proof.

(Sketch): expand each model M of T by a new sort $X(M)$. Use commutation with ultraproducts to verify this is an elementary class. Then we are in the situation of $1 \implies (*)$ from Beth's theorem. ∎

How does strong conceptual completeness enter this picture?

- Plain old conceptual completeness (this was one of the key results of Makkai-Reyes) says that if an interpretation $I : T_1 \to T_2$ induces an equivalence of categories $\mathbf{Mod}(T_1) \overset{I^*}{\simeq} \mathbf{Mod}(T_2)$, then I must have been a bi-interpretation.
  So, it proves $1 \implies (*)$, and therefore the corollary.
- Strong conceptual completeness is the following upgrade of the corollary.

Strong conceptual completeness, I

Theorem.

Let T be an L-theory. Let X be any functor $\mathbf{Mod}(T) \to \mathbf{Set}$. Suppose that you have:

- *for every ultraproduct $\prod_{i \to \mathcal{U}} M_i$ a way to identify $X(\prod_{i \to \mathcal{U}} M_i) \overset{\Phi_{(M_i)}}{\simeq} \prod_{i \to \mathcal{U}} X(M_i)$ ("there exists a transition isomorphism"), such that*
- *(X, Φ) preserves ultraproducts of models/elementary embeddings ("is a pre-ultrafunctor"), and also*
- *preserves all canonical maps between ultraproducts ("preserves ultramorphisms").*

Then: there exists a $\varphi(x) \in T^{\mathrm{eq}}$ such that $X \simeq \mathrm{ev}_{\varphi(x)}$ as functors $\mathbf{Mod}(T) \to \mathbf{Set}$. (We call such X an ultrafunctor.)

Strong conceptual completeness, I

- That is, the specified transition isomorphisms $\Phi_{(M_i)} : X\left(\prod_{i \to \mathcal{U}} M_i\right) \to \prod_{i \to \mathcal{U}} X(M_i)$ make all diagrams of the form

$$\begin{array}{ccc} X\left(\prod_{i \to \mathcal{U}} M_i\right) & \xrightarrow{\Phi_{(M_i)}} & \prod_{i \to \mathcal{U}} X(M_i) \\ {\scriptstyle X\left(\prod_{i \to \mathcal{U}} f_i\right)}\downarrow & & \downarrow{\scriptstyle \prod_{i \to \mathcal{U}} X(f_i)} \\ X\left(\prod_{i \to \mathcal{U}} N_i\right) & \xrightarrow[\Phi_{(N_i)}]{} & \prod_{i \to \mathcal{U}} X(N_i) \end{array}$$

  commute ("transition isomorphism/pre-ultrafunctor condition").

Strong conceptual completeness, I

What are ultramorphisms?

An **ultragraph** Γ comprises:

- A directed graph whose vertices are partitioned into *free nodes* Γ^f and *bound nodes* Γ^b.
- For any bound node $\beta \in \Gamma^b$, we assign a triple $\langle I, \mathcal{U}, g \rangle \overset{\text{df}}{=} \langle I_\beta, \mathcal{U}_\beta, g_\beta \rangle$ where $\mathcal{U}$ is an ultrafilter on I and g is a function $g : I \to \Gamma^f$.
- An ultradiagram for Γ is a diagram of shape Γ which incorporates the extra data: bound nodes are the ultraproducts of the free nodes given by the functions g.
- A *morphism* of ultradiagrams (for fixed Γ) is just a natural transformation of functors which respects the extra data: the component of the transformation at a bound node is the ultraproduct of the components for the indexing free nodes.

Strong conceptual completeness, I

Okay, but *what are ultramorphisms*?

Definition.

Let $\mathrm{Hom}(\Gamma, \underline{\mathbf{S}})$ *be the category of all ultradiagrams of type* Γ *inside* $\underline{\mathbf{S}}$ *with morphisms the ultradiagram morphisms defined above. Any two nodes* $k, \ell \in \Gamma$ *define evaluation functors* $(k), (\ell) : \mathrm{Hom}(\Gamma, \underline{\mathbf{S}}) \rightrightarrows \mathbf{S}$, *by*

$$(k)\left(A \overset{\Phi}{\to} B\right) = A(k) \overset{\Phi_k}{\to} B(k)$$

(resp. ℓ*).*
An **ultramorphism** *of type* $\langle \Gamma, k, \ell \rangle$ *in* $\underline{\mathbf{S}}$ *is a natural transformation* $\delta : (k) \to (\ell)$.

It's sufficient to consider the ultramorphisms which come from universal properties of colimits of products in **Set**.

Strong conceptual completeness, II

Now, what's changed between this statement and that of the useful corollary to Beth's theorem?

- We dropped the *subfunctor* assumption! We don't have such a nice way of knowing exactly how $X(M)$ is obtained from M. We only have the invariance under ultra-stuff. We've left the placental warmth of the ambient models and we're considering some kind of abstract permutation representation of $\mathbf{Mod}(T)$.
- Yet, if X respects enough of the structure induced by the ultra-stuff, then X must have been constructible from our models in some first-order way ("is definable").
- (With this new language, the corollary becomes: "strict sub-pre-ultrafunctors of definable functors are definable.")

Strong conceptual completeness, III

Actually, Makkai proved something more, by doing the following:

- Introduce the notions of ultracategory and ultrafunctors by requiring all this extra ultra-stuff to be preserved.
- Develop a general duality theory between pretoposes ("$\mathbf{Def}(T)$") and ultracategories ("$\mathbf{Mod}(T)$") via a contravariant 2-adjunction ("generalized Stone duality").
- In particular, from this adjunction we get $\mathbf{Pretop}(T_1, T_2) \simeq \mathbf{Ult}(\mathbf{Mod}(T_2), \mathbf{Mod}(T_1))$.

Therefore, SCC tells us how to recognize a reduct functor in the wild between two categories of models—i.e., if there is some uniformity underlying a functor $\mathbf{Mod}(T_2) \to \mathbf{Mod}(T_1)$ due to a purely syntactic assignment $T_1 \to T_2$. Just check if the ultra-structure is preserved!

Caveat. Of course, one has an infinite list of conditions to verify here.

- So the only way to actually do this is to recognize some kind of uniformity in the putative reduct functor which lets you take care of all the ultramorphisms at once.
- But it gives you another way to think about uniformities you need.
- It also gives you a way to check that something can never arise from any interpretation!

Important examples of ultramorphisms

Examples.

- *The diagonal embedding into an ultrapower.*
- *Generalized diagonal embeddings. More generally, let $f : I \to J$ be a function, let $\mathcal{U}$ be an ultrafilter on I and let $\mathcal{V}$ be the pushforward ultrafilter on J. Then for any I-indexed sequence of structures $(M_i)_{i \in I}$, there is a canonical map $\delta_f : \prod_{j \to \mathcal{V}} M_{f(i)} \to \prod_{i \to \mathcal{U}} M_i$ given by taking the diagonal embedding along each fiber of f.*

Δ-functors induce continuous maps on automorphism groups

- Why should we expect ultramorphisms to help us identify evaluation functors in the wild?
- Here's an result which might indicate that knowing that they're preserved tells us something nontrivial.

Definition.

Say that $X : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ *is a* Δ*-functor if it preserves ultraproducts and diagonal maps into ultrapowers.*

Equip automorphism groups with the topology of pointwise convergence.

Theorem.

If X *is a* Δ*-functor from* $\mathbf{Mod}(T)$ *to* $\mathbf{Mod}(T')$*, then* X *restricts to a continuous map* $\mathrm{Aut}(M) \to \mathrm{Aut}(X(M))$ *for every* $M \in \mathbf{Mod}(T)$*.*

Proof.

- The topology of pointwise convergence is sequential, so to check continuity it suffices to check convergent sequences of automorphisms are preserved.
- If $f_i \to f$ in $\mathrm{Aut}(M)$, then since the cofinite filter is contained in any ultrafilter, $\prod_{i \to \mathcal{U}} f_i$ agrees with $\prod_{i \to \mathcal{U}} f$ over the diagonal copy of M in $M^{\mathcal{U}}$. That is, $(\prod_{i \to \mathcal{U}} f_i) \circ \Delta_M = (\prod_{i \to \mathcal{U}} f) \circ \Delta_M$.
- Applying X and using that X is a Δ-functor, conclude that $\prod_{i \to \mathcal{U}} X(f_i)$ agrees with $\prod_{i \to \mathcal{U}} X(f)$ over the diagonal copy of $X(M)$ inside $X(M)^{\mathcal{U}}$.
- For any point $a \in X(M)$, the above says the sequence $(X(f_i)(a))_{i \in I} =_{\mathcal{U}} (X(f)(a))_{i \in I}$.
- Since $\mathcal{U}$ was arbitrary and the cofinite filter on I is the intersection of all non-principal ultrafilters on I, we conclude that the above equation holds cofinitely. Hence, $X(f_i) \to X(f)$.

□

$\aleph_0$-categorical theories

- A first-order theory T is $\aleph_0$-categorical if it has one countable model up to isomorphism.
- $\aleph_0$-categorical theories have only finitely many types in each sort. (Caveat: when I say "type", I mean an atom in $\mathscr{E}(T)$.)
- A theorem of Coquand, Ahlbrandt and Ziegler says that, given two $\aleph_0$-categorical theories T and T' with countable models M and M', a topological isomorphism $\text{Aut}(M) \simeq \text{Aut}(M')$ induces a bi-interpretation $M \simeq M'$.
- Since we know Δ-functors induce continuous maps on automorphism groups, they're a good candidate for definable functors.
- Boolean coherent toposes split into a finite coproduct of $\mathscr{E}(T_i)$, where each T_i *is* $\aleph_0$-categorical.

A definability criterion for $\aleph_0$-categorical theories

Theorem.

Let $X : \mathbf{Mod}(T) \to \mathbf{Set}$. If T is $\aleph_0$-categorical, the following are equivalent:

1. *For some transition isomorphism, (X, Φ) is a Δ-functor (preserves ultraproducts and diagonal maps).*
2. *For some transition isomorphism, (X, Φ) is definable.*

A definability criterion for $\aleph_0$-categorical theories

Proof.

(Sketch.)

- One direction is immediate by SCC: definable functors are ultrafunctors are at least Δ-functors.
- Let M be the countable model. Use the lemma about Δ-functors (X, Φ) inducing continuous maps on the automorphism groups (equivalently, (X, Φ) has the finite support property) to cover each $\text{Aut}(M)$-orbit of $X(M)$ by a projection from an $\text{Aut}(M)$-orbit of M. By ω-categoricity, the kernel relation of this projection is definable, so we know that $X(M)$ looks like an (*a priori*, possibly infinite) disjoint union of types.
- By $\text{Aut}(M)^{\mathcal{U}}$ orbit-counting, there are actually only finitely many types.
- Invoke the Keisler-Shelah theorem to transfer to all $N \models T$.

A definability criterion for $\aleph_0$-categorical theories

Corollary.

Let T and T' be $\aleph_0$-categorical. Let X be an equivalence of categories

$$\mathbf{Mod}(T_1) \overset{X}{\simeq} \mathbf{Mod}(T_2).$$

Then X was induced by a bi-interpretation $T_1 \simeq T_2$ if and only if X was a Δ-functor.

In particular, Bodirsky, Evans, Kompatscher and Pinkser gave an example of two $\aleph_0$-categorical theories T, T' with abstractly isomorphic but not topologically isomorphic automorphism groups of the countable model. This abstract isomorphism induces an equivalence $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ and since it can't come from an interpretation, from the corollary we conclude that it fails to preserve an ultraproduct or a diagonal map was not preserved.

Exotic pre-ultrafunctors

In light of the previous result, a natural question to ask is:

Question.

Is being a Δ-functor enough for SCC? That is, do non-definable Δ-functors exist?

Theorem.

The previous definability criterion fails for general T. That is:

- *There exists a theory T and a Δ-functor $(X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set}$ which is not definable.*
- *There exists a theory T and a pre-ultrafunctor (X, Φ) which is not a Δ-functor (hence, is also not definable.)*

Exotic pre-ultrafunctors

Proof.

(Sketch.)

- Complete types won't work, so take a complete type and cut it in half into two partial types, one of which refines the other. Define $X(M)$ to be the realizations in M of the coarser one.
- Taking ultraproducts creates external realizations ("infinite/infinitesimal points") of either one.
- You can either try to construct a transition isomorphism which turns it into a pre-ultrafunctor (creating a non-Δ pre-ultrafunctor) or obtain one non-constructively (creating a non-definable Δ-functor).

□

Future work

- Is the above $X(M)$ isomorphic to ev_A for some $A \in \mathscr{E}(T)$?
- Which parts of Makkai's ultra-data ensure $X : \mathbf{Mod}(T) \to \mathbf{Set}$ is ev_A for $A \in \mathscr{E}$ and which parts make sure that A is compact?
- How do ultramorphisms relate to the Awodey-Forssell duality?
- Conjecture: the pre-ultrafunctor part of the data ensures compactness after you get inside the classifying topos, i.e. if you start with $A \in \mathscr{E}$ and ev_A is an ultrafunctor, then A was compact.
- **Update:** this last conjecture is actually true!

Latest results:

Theorem.

Let $\mathscr{E}(T)$ be the classifying topos of a first-order theory. Let B be an object of $\mathscr{E}(T)$. The following are equivalent:

1. *B is coherent.*
2. $\text{ev}_B : \mathbf{Mod}(T) \to \mathbf{Set}$ *is a pre-ultrafunctor.*
3. *The reduct functor $\mathbf{Mod}(T[B]) \xrightarrow{I^*} \mathbf{Mod}(T)$ is an equivalence, where $T[B]$ is T with an additional sort for B and all the induced definable structure on B ("the graph of $\mathscr{E}(T)(\mathbf{y}(-), B)$") adjoined.*
4. *$\mathbf{Mod}(\mathscr{E}(T)/B)$ is an ultracategory such that the forgetful functor $F : \mathbf{Mod}(\mathscr{E}(T)/B) \to \mathbf{Mod}(T)$ is an ultrafunctor and the functor $(\langle M, b\rangle \mapsto \{b\}) : \mathbf{Mod}(\mathscr{E}(T)/B) \to \mathbf{Set}$ is a strict ultrafunctor.*

Thank you!